THÈSE

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Comportement en temps long de différents processus stochastiques en neuroscience

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Don't let anyone rob you of your imagination, your creativity, or your curiosity. It's your place in the world; it's your life.

Mae Jemison,
first African American woman
astronaut in space (1992)

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Contents

Chapitre 1
Introduction

Cette thèse porte sur deux processus stochastiques, les processus de Hawkes et les processus de FitzHugh-Nagumo. Ceux-ci, particulièrement distincts comme nous le verrons dans la suite, peuvent tous deux servir à de la modélisation. Là où les processus de Hawkes sont applicables à des phénomènes très divers, comme les séismes [43, 44], les réseaux sociaux [34, 32], la finance [42, 3], la biologie (génomique [73], neurologie [58, 41]) ; les processus de FitzHugh-Nagumo ont été spécifiquement construits pour une application : les décharges électriques des neurones. Nous commençons donc par introduire, en quelques pages, la modélisation des neurones dans le cerveau, afin de comprendre comment deux processus si distincts peuvent être intéressants dans ce cadre.
Ensuite nous décrirons plus explicitement ces processus, le contexte dans lequel nous les avons étudiés et les résultats obtenus.

1.1 Quelques explications sur les neurones

1.1.1 Préambule biologique

Le fonctionnement biologique du cerveau et des neurones est encore très peu connu. Différentes approches, et différents modèles existent afin de, petit à petit, comprendre certaines spécificités et certains fonctionnements. Notre point de vue ici, via les modèles que nous étudions, est un point de vue microscopique : l'idée est d'essayer de comprendre comment fonctionne un neurone, puis plusieurs neurones pour comprendre le fonctionnement général d'un ensemble de neurones. Il ne s'agit bien sûr pas de modéliser un cerveau entier : un humain a de l'ordre de 10^{11} neurones, ce qui est immense, et ne peut donc pas être étudié en considérant les neurones un à un. D'autres types de modèles, dits macroscopiques, existent pour ces questions. Des liens existent entre modèles microscopiques et modèles macroscopiques, notamment via ce que nous avons étudié dans un cadre particulier avec la propagation du chaos.

Tout d'abord, un neurone est une cellule. Cette dernière est spécifique et permet de transmettre des signaux électriques sur des distances pouvant être grandes. Il est composé d'un *soma* (le corps de la cellule), d'un *axone* et d'un ou plusieurs *dendrites*, comme le décrit la figure 1.1. Les dendrites permettent de réceptionner les signaux émis par d'autres neurones. L'axone, unique lui, a plusieurs terminaisons axonales qui permettent au neurone de transmettre un signal.

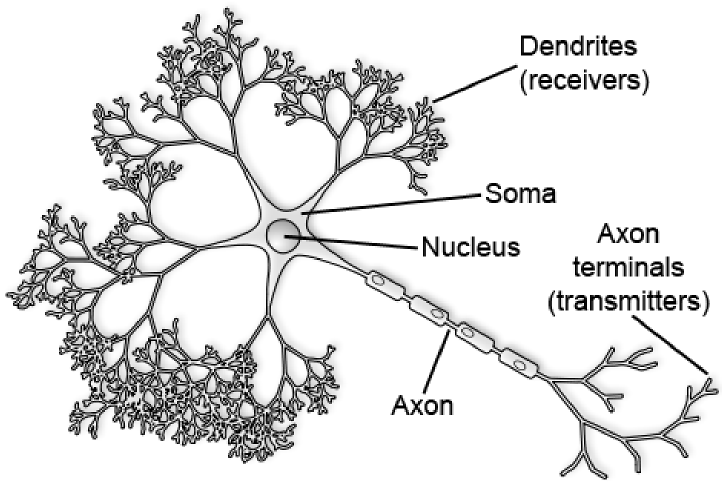


Figure 1.1 – Représentation d'un neurone, par Nicolas Rougier (2007) [1]

Les signaux sont transmis par des *synapses*. Ces synapses sont les points de jonction entre deux neurones, généralement entre un axone et une dendrite. Deux types de synapses existent : les synapses chimiques et les synapses électriques.
Dans la synapse chimique, le signal nerveux est transmis par une molécule spécifique appelée neurotransmetteur. Celle-ci est émise par le neurone qui envoie le signal et se lie à des récepteurs sur le second neurone.
Dans la synapse électrique, l'influx nerveux se transmet par courant ionique : les ions se déplacent d'un neurone à un autre.
Ces deux types de synapses ont chacune des caractéristiques propres et des modélisations distinctes.

Un neurone typique reçoit des signaux venant de plusieurs milliers de neurones. Chacune de ces contributions entraîne un changement du *potentiel de membrane* du neurone. Ces apports peuvent être d'amplitude variée et entraînent un changement de potentiel proportionnel. Le neurone va ensuite générer une *décharge* électrique qui est, par la suite, propagée le long de son axone en direction des autres neurones. Les déclenchements de décharges ne sont pas encore entièrement compris. L'idée classique est de considérer un seuil : tant que le potentiel du neurone est en dessous du seuil, il ne se passe rien, et lorsqu'il le dépasse, une décharge est produite avec la totalité du potentiel. Cette idée est mise à mal par plusieurs expériences. En effet, à plusieurs reprises, des biologistes ont tenté de déterminer le seuil de déclenchement pour certains types de neurones, mais ils n'ont pas pu l'identifier. En fait, selon la durée du signal reçu par le neurone (instantané, ou sur une durée plus longue), son type (signal unique ou répétitif), sa période (dans le cas où il est périodique), le seuil de déclenchement peut être différent. Il ne s'agit pas simplement d'une condition en

1. Nicolas Rougier (2007) Wikimedia Commons. (`https://commons.wikimedia.org/wiki/File:Neuron-figure-fr.svg`) CC BY 4.0

tout-ou-rien : "au-delà de telle valeur" et "en dessous de telle valeur".
Différents modèles mathématiques ont donc été construits pour étudier la transmission de ces signaux électriques entre les neurones. Parmi ces modèles, certains considèrent un seuil, et d'autres ont des interactions plus nuancées.

Les questions des biologistes sont extrêmement variées. Ainsi, le choix du modèle dépend des points d'intérêts, du type de données fournies et de la précision souhaitée. Les données obtenues sont souvent de la forme de *spiking train*, c'est-à-dire d'une suite de temps de décharge de neurones. Il n'est d'ailleurs pas toujours possible d'isoler les temps de décharge pour un neurone par rapport aux autres d'un point de vue expérimental, et des outils statistiques sont donc développés pour le faire.
Le taux de décharge, ou *firing rate*, est une quantité souvent étudiée dans les articles. Il s'agit de l'inverse du nombre de décharges sur une certaine durée, qui permet de calibrer le modèle à partir de vraies données mais qui est également une quantité intéressante en soi.

1.1.2 Les modèles

Lorsque l'on s'intéresse aux comportements des neurones, deux types de modèles existent : les modèles à sauts, dans lesquels on s'intéresse aux instants de décharge du neurone (et éventuellement à leur amplitude), et les modèles continus, dans lesquels on considère l'évolution du potentiel du neurone.
La construction et l'utilisation de ces deux types de modèles dans les neurosciences ont été faites de manière différente : là où les modèles continus ont en général été construits à partir des expériences et d'hypothèses, et sont spécifiques aux neurones ; les modèles à sauts ont pu être construits et étudiés pour d'autres objectifs, puis ont été appliqués aux neurones.

1.1.2.1 Modèles à sauts

Lorsque l'on regarde une suite d'instants aléatoires, le plus naturel est de considérer un processus de saut. Ces processus, aussi appelés processus ponctuels sur $\mathbb{R}$, sont représentés par une suite de temps $(S_i)_{i\in\mathbb{N}}$ ou de manière équivalente par une suite d'intervalles $(\tau_i)_{i\in\mathbb{N}^*} = (S_i - S_{i-1})_{i\in\mathbb{N}^*}$. Ils sont caractérisés par une valeur initiale S_0, souvent nulle, et par une loi pour les τ_i.

Le processus le plus simple est bien entendu le processus de Poisson homogène : la suite $(\tau_i)_{i\in\mathbb{N}^*}$ est i.i.d. et τ_1 suit une loi exponentielle de paramètre constant λ. On dit alors que l'intensité du processus Λ est constante et vaut λ. Dans le cadre de modélisation de neurones, ce processus est trop simple pour être utilisé : par exemple, il ne permet pas de représenter des périodes réfractaires (périodes qui suivent un déclenchement de décharge et pendant lesquelles le neurone ne peut plus se redéclencher). Une seconde limite, assez générale pour les processus ponctuels, est que le processus de Poisson homogène est stationnaire, et il a déjà été montré, par exemple dans [75], que les décharges neuronales ne peuvent pas être entièrement modélisées par un processus stationnaire.

Une manière de complexifier cette idée est de considérer les processus de renouvellement, comme dans [67]. Ces processus sont décrits par une suite $(\tau_i)_{i\in\mathbb{N}^*}$ qui est i.i.d. mais dont la loi n'est pas nécessairement exponentielle. Dans [67], on suppose que τ_1 suit une loi Gamma ou une loi Log-Normale. Cependant ce modèle présente la même limite que le modèle de Poisson homogène : le processus est stationnaire. Il s'agit néanmoins d'un modèle très simple qui peut être utilisé pour certaines études. Pour aller plus loin, [68] et [21] ont considéré des processus de Wold, c'est-à-dire des processus dont le futur saut dépend du dernier instant de saut, mais également de la longueur du dernier intervalle : en fait τ_{n+1} dépend de τ_n.

Une seconde manière de complexifier l'idée d'un processus de Poisson homogène est de s'intéresser à des processus de Poisson inhomogènes, dont l'intensité Λ varie au cours du temps. C'est le type de processus considéré dans [83] et dans [72]. Néanmoins, [39] et [66] ont mis en évidence une dépendance des décharges des neurones entre elles, ce qui ne peut pas être modélisé par ces processus. Les processus de Hawkes permettent d'étendre la notion de processus de Poisson en considérant une dépendance des évènements entre eux, c'est pourquoi [72] et [21] s'y sont intéressés dans le cadre de la modélisation de neurones. Même si les processus de Hawkes sont également stationnaires, ils permettent tout de même de modéliser dans une certaine mesure des décharges neuronales. Ce processus que nous avons choisi d'étudier est défini plus en détail dans la suite.

1.1.2.2 Modèle continu

Pour étudier l'évolution du potentiel d'un neurone, différents modèles ont été décrits avec des équations différentielles ordinaires (EDO). Ici, nous n'en évoquerons que quelques-uns, en décrivant en particulier les équations différentielles stochastiques (EDS) qui en découlent. Pour avoir plus d'informations et d'explications biochimiques sur les modèles, le livre [48] présente les différents systèmes dynamiques (déterministes) utilisés en neuroscience.
De manière générale, les systèmes que nous présentons ici découlent tous de la représentation d'une synapse comme d'un circuit électrique, et de l'équation

$$C\frac{dX(t)}{dt} = I_{input}(t) - I_{model}(t),$$

où X est la différence de potentiel, C est la capacité, I_{input} est une intensité en entrée du système (avec des électrodes), et I_{model} est l'intensité dans le neurone décrite par le modèle.

Le premier d'entre eux, attribué à Lapique en 1907 (bien qu'il ait en fait été nommé et décrit ainsi à partir des années 1960 d'après [17]), est le modèle Intégration-Et-Tire (Integrate-and-fire). Dans ce modèle, $I_{model}(t) = 0$ tant que $X(t)$ n'a pas atteint une certaine valeur seuil X_{th}. Lorsque $X(t)$ atteint cette valeur, on considère que le neurone a une décharge et que $X(t)$ revient à 0. Des modifications de ce modèle ont été proposées, dès 1907, notamment le modèle Intégration et Tire avec fuite (Leaky integrate-and-fire) qui prend en compte une fuite (ou diffusion) des ions à travers la membrane du neurone. Dans ce cas, $I_{model}(t)$ est proportionnel à $X(t)$. En particulier, une version stochastique de ce modèle existe et est étudié, notamment dans [49]. Dans cet article, les paramètres sont estimés à partir des données expérimentales, et le modèle semble bien adapté.

Néanmoins, ce modèle est descriptif et ne permet pas de comprendre la manière dont les décharges se produisent. Hodgkin et Huxley [45], en 1952, ont travaillé sur l'axone géant du calmar pour comprendre plus spécifiquement le fonctionnement du neurone. Ils ont décrit le courant du modèle en le décomposant en trois courants ioniques : $I_{model} = I_{\mathrm{Na}} + I_{\mathrm{K}} + I_l$. Les ions sodium et potassium représentent la part principale de ce courant, et I_l est le courant "de fuite", généré par les ions chlorure et d'autres ions. I_{K} et I_{Na} sont décrits à partir de X et de trois variables n, m et h qui sont les variables d'activation ou de désactivation liées aux canaux ioniques. Pour chacune de ces variables, une EDO permet de décrire son comportement (en fonction d'elle-même et de X). Ce modèle est donc constitué de quatre EDO pour un seul neurone. Différentes versions stochastiques de ce modèle ont été étudiées. Ainsi, [46] considère un courant d'entrée I_{input} stochastique, tandis que [4] considère un bruit brownien sur les différentes équations. Ce dernier article, consacré à l'étude de ce modèle dans le cadre de plusieurs neurones en interaction champ-moyen, est complété par [12].

Le modèle de Hodgkin-Huxley est reconnu comme étant très complet, mais en raison de sa complexité (chaque neurone est décrit par 4 EDO ou EDS non-linéaires), plusieurs modèles simplifiés ont été proposés et étudiés, en règle générale en se réduisant à 2 EDO ou EDS. Certains modèles se sont concentrés sur les courants de certains ions, comme le modèle de Morris-Lecar [64], aussi appelé modèle $I_{\mathrm{Ca}} + I_{\mathrm{K}}$. Une version stochastique a été décrite et [26] décrit une méthode pour estimer des paramètres physiologiques à partir de ce modèle et de données partielles obtenues par l'expérience.
Une autre simplification possible de ce modèle est le modèle de FitzHugh-Nagumo, sur lequel nous nous sommes concentrés dans cette thèse. Celui-ci a été défini en 1961 par FitzHugh [33] et le circuit électrique équivalent a été décrit par Nagumo, Arimoto et Yoshizawa en 1962 [65].

1.2 Plan de la thèse et chaque chapitre

Pendant cette thèse, je me suis concentrée sur les deux modèles précédemment cités : le modèle de Hawkes et le modèle de FitzHugh-Nagumo.

Tout d'abord, j'ai souhaité généraliser des résultats connus pour une sous-catégorie de processus de Hawkes : le cas des processus dit linéaires et auto-excitants. Je me suis intéressée à des processus qui pouvaient représenter de l'auto-inhibition ou une auto-influence à la fois inhibitrice et excitatrice. L'objectif est de permettre la modélisation de phénomènes plus complexes : il est par exemple connu que les neurones peuvent être inhibés. On peut également considérer une auto-influence mixte, d'abord inhibante - ce qui empêche l'arrivée d'un second évènement trop tôt après un premier - puis excitante après une certaine durée. Je me suis restreinte au cas où cette auto-influence est sur une durée limitée. Cette restriction est cohérente avec les applications possibles, et permet de représenter les processus de Hawkes comme des processus cumulatifs, qui sont un type de processus de renouvellement plus général. Cette représentation m'a permis de démontrer une loi des grands nombres et un théorème central limite, qui était déjà connus pour les processus cumulatifs, dans mon cadre. Le Chapitre 3 décrit ce travail.
J'ai souhaité compléter mon étude avec un Principe de Grandes Déviations, ou a minima des inégalités de grandes déviations. Néanmoins, ce principe n'existait pas pour les processus cumulatifs dans la littérature dans le cadre où je souhaitais l'appliquer. J'ai donc démontré un Principe de Grandes Déviations, et des inégalités de grandes déviations sous certaines hypothèses : cette étude est détaillée dans le Chapitre 2. J'ai ensuite pu appliquer ce Théorème dans le cadre des processus de Hawkes, ce qui achève le Chapitre 3.
Pour cette raison, je présente de manière liée ces deux Chapitres dans la Sous-Section 1.2.1 suivante.

Ensuite, je me suis intéressée au modèle de FitzHugh-Nagumo stochastique. Alors que, pour les processus de Hawkes, je me focalisais sur un seul processus, ici j'ai souhaité considérer des processus en interaction, pouvant représenter plusieurs neurones qui communiquent. Je me suis concentrée sur le cadre d'une interaction champ-moyen, et j'ai montré une propagation du chaos, d'abord non-uniforme en temps, puis dans un second temps, uniforme en temps. Pour ce faire, j'ai collaboré avec Pierre Le Bris, et nous avons utilisé une méthode de couplage mixte, c'est-à-dire un couplage synchrone sur un certain sous-espace, et un couplage symétrique sur l'espace complémentaire. Cette approche est l'objet du Chapitre 4.

1.2.1 Processus à sauts - Chapitres 2 et 3

1.2.1.1 Définitions et propriétés des processus cumulatifs

Les processus cumulatifs (cumulative process) ont été décrits par Smith [76] et sont appliqués dans de nombreux contextes, comme en finance où ils sont appelés processus de renouvellement composé (*compound-renewal* processes) ou processus de renouvellement à récompense (*renewal-reward* processes). Ces processus cumulent des variables aléatoires indépendantes au cours du temps (continu). Ces variables aléatoires sont ajoutées sur des intervalles de temps donnés par un processus de renouvellement.

Nous considérons un espace de probabilité muni d'une filtration appropriée $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ qui satisfait les hypothèses habituelles.

Définition 1.2.1. Un processus $(Z_t)_t : \mathbb{R}^+ \to \mathbb{R}$ est dit *cumulatif* s'il vérifie les propriétés suivantes :

1. $Z_0 = 0$,
2. il existe un processus de renouvellement $(S_n)_n$ tel que pour tout n, $(Z_{S_n+t} - Z_{S_n})_{t\geq 0}$ est indépendant de $S_0, ...S_n$ et de $(Z_s)_{s<S_n}$,
3. la distribution de $(Z_{S_n+t} - Z_{S_n})_{t\geq 0}$ est indépendante de n.

Néanmoins, travailler avec cette définition n'est pas toujours simple. On préfère en général travailler avec l'écriture

$$Z_t = W_0(t) + \sum_{i=1}^{M_t} W_i + r_t, \tag{1.2.1}$$

où :
- $(\tau_i)_i$ est la suite de durées associée au processus de renouvellement $(S_n)_n$;
- $M_t = \sup\{n \geq 0, S_n \leq t\}$, le processus de comptage associé à $(S_n)_n$;
- $W_i = Z_{S_n} - Z_{S_{n-1}}$;
- $W_0(t) = Z_{t\wedge\tau_1} = Z_{\inf(t,\tau_1)}$;
- $r_t = Z_t - Z_{M_t}$.

En particulier, on a alors $(\tau_1, W_1), (\tau_2, W_2), \ldots$ une suite i.i.d. de couples de variables aléatoires dans $\mathbb{R}^+ \times \mathbb{R}$.

Remarque 1.2.2. On peut définir un exemple de processus cumulatif de la façon suivante : soit une suite i.i.d. de couples de variables aléatoires $(\tau_1, W_1), (\tau_2, W_2), \ldots$ dans $\mathbb{R}^+ \times \mathbb{R}$, définies sur un espace de probabilité $(\Omega, \mathbb{P})$. Nous considérons le processus de renouvellement associé à $(\tau_i)_{i\geq 1}$

$$S_0 = 0, \;\; S_n = \sum_{k=1}^{n} \tau_k, \quad \text{et} \quad M_t = \sup\{n \geq 0, S_n \leq t\}\,.$$

La variable aléatoire

$$Z_t = \sum_{i=1}^{M_t} W_i$$

est un processus cumulatif, avec $W_0 = 0$ et $r_t = 0$.

Il est important de noter qu'en général, à i fixé, τ_i et W_i ne sont pas indépendants. Comme les couples $(\tau_i, W_i)_i$ sont i.i.d., on supprimera dans la suite l'indice i lorsque l'on traitera de la distribution (et des quantités associées comme l'espérance et la variance) de (τ_i, W_i) et nous écrirons simplement (τ, W). De façon évidente, nous supposons $\mathbb{E}(\tau) > 0$.

La loi des grands nombres (en supposant $\mathbb{E}[|W|]$ et $\mathbb{E}[\tau]$ finis)

$$\frac{Z_t}{t} \xrightarrow[t\to\infty]{p.s.} \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]} \text{ si et seulement si } \mathbb{E}\left(\max_{S_0\le t<S_1} |r_t|\right) < \infty\,,$$

et le théorème central limite (en supposant $\mathrm{Var}(W) < \infty$ et $\mathrm{Var}(\tau) < \infty$)

$$\frac{\left(Z_t - t\frac{\mathbb{E}[W]}{\mathbb{E}[\tau]}\right)}{\sqrt{t}} \xrightarrow[t\to\infty]{} \mathcal{N}\left(0,\sigma^2\right) \text{ où } \sigma^2 = \frac{1}{\mathbb{E}(\tau)}\mathrm{Var}\left(W - \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]}\tau\right)$$

peuvent être trouvés dans le livre d'Asmussen [1], théorème 3.1 et théorème 3.2.

L'objectif de notre travail est d'établir un principe de grande déviation pour les processus cumulatifs. Certaines études ont déjà été effectuées dans ce sens. En particulier, Duffy et Metcalfe [28] ont travaillé sur l'estimation d'une fonction de taux pour un processus cumulatif (s'il admet un principe de grande déviation). Dans une suite d'articles, Borovkov et Mogulskii ([9, 10, 11]) ont également étudié le principe de grande déviation, sous certaines hypothèses de comparaison entre la loi jointe de (τ, W) et la loi de τ. Certains détails ne nous semblaient pas clairs, et nous avons souhaité établir un principe dans un cadre différent, l'objectif étant de pouvoir l'appliquer aux processus de Hawkes.
Lefevere, Mariani et Zambotti [52] ont travaillé sur des processus cumulatifs spécifiques pour lesquels $W_i = F(\tau_i)$, pour F une fonction déterministe qui est supposée positive, bornée et continue. Nous avons choisi de suivre leur approche.

1.2.1.2 Chapitre 2 : Grandes déviations pour les processus cumulatifs

Principe de grandes déviations - rappels généraux

L'objectif de ce chapitre est de prouver un Principe de Grandes Déviations pour le processus Z_t/t. Rappelons quelques définitions sur les grandes déviations (tirées de [25]).

Une famille de mesures de probabilité $(\eta_t)_{t\in\mathbb{R}^+}$ sur un espace topologique $(\mathcal{X}, T_{\mathcal{X}})$ satisfait un principe de grandes déviations avec la fonction de taux $J(.)$ et la vitesse $\gamma(t) = t$ si $J : \mathcal{X} \to [0, +\infty]$ est semi-continue inférieurement et satisfait

$$-\inf_{x\in\mathcal{O}} J(x) \le \liminf_{t\to+\infty} \frac{1}{t} \ln \eta_t(\mathcal{O}) \quad \text{pour tout ouvert } \mathcal{O}, \tag{1.2.2}$$

et

$$-\inf_{x\in\mathcal{C}} J(x) \ge \limsup_{t\to+\infty} \frac{1}{t} \ln \eta_t(\mathcal{C}) \quad \text{pour tout fermé } \mathcal{C}. \tag{1.2.3}$$

Nous pourrons dire que $(\eta_t)_{t\in\mathbb{R}^+}$ satisfait un Principe de Grandes Déviations *complet* si (1.2.2) et (1.2.3) sont satisfaits, tandis que nous parlerons de Principe de Grandes Déviations *faible* quand les fermés $\mathcal{C}$ sont remplacés par des compacts $\mathcal{C}$

dans (1.2.3). Quand η_t est la distribution d'une variable aléatoire Y_t (par exemple Z_t/t) nous dirons que la famille $(Y_t)_t$ satisfait un Principe de Grandes Déviations.

Puisque la fonction de taux J est semi-continue inférieurement, les ensembles de niveau $\{X, J(x) \leq a\}$ sont fermés. S'ils sont également compacts, alors J sera qualifiée de bonne fonction de taux.

Nous considérerons ici uniquement la vitesse $\gamma(t) = t$, nous n'y ferons donc plus référence.

Une notion particulièrement importante pour notre objectif est la notion d'*approximation exponentiellement bonne.*

Définition 1.2.3. Supposons $(\mathcal{X}, d)$ est un espace métrique. Une famille de variables aléatoires $Y_{n,t}$ $(n \in \mathbb{N})$ est une approximation exponentiellement bonne de Y_t (toutes ces variables sont supposées définies sur le même espace de probabilité $(\Omega, \mathbb{P})$), si pour tout $\delta > 0$ on a

$$\lim_{n\to\infty} \limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}(d(Y_{n,t}, Y_t) > \delta) \; = \; -\infty \, .$$

Le résultat clé pour notre étude est

Théorème 1.2.4. *Dans le cadre de la définition 1.2.3, on suppose que $Y_{n,t}$ est une approximation exponentiellement bonne de Y_t. Alors, les propositions suivantes sont vraies*

1. *Si $Y_{n,t}$ satisfait un Principe de Grandes Déviations complet, avec pour fonction de taux J^n, alors Y_t satisfait un Principe de Grandes Déviations faible avec la fonction de taux*
$$J(x) = \sup_{\delta>0} \liminf_{n\to\infty} \inf_{d(y,x)<\delta} J^n(y) \, .$$
2. *Si $\mathcal{X} = \mathbb{R}^k$ équipé d'une norme quelconque, alors la même conclusion reste vraie lorsque $Y_{n,t}$ satisfait uniquement un Principe de Grandes Déviations faible.*
3. *Si J (définie ci-dessus) est une bonne fonction de taux, telle que pour tout fermé F,*
$$\inf_{y\in F} J(y) \leq \limsup_{n\to\infty} \inf_{y\in F} J^n(y)$$
alors Y_t satisfait un Principe de Grandes Déviations complet, avec la fonction de taux J.

Le premier et le dernier de ces points sont contenus dans le Théorème 4.2.16 de [25]. Le second point est une conséquence du fait que les boules fermées sont des compacts dans $\mathbb{R}^k$.
Habituellement, ce Théorème est suffisant pour prouver un Principe de Grandes

Déviations complet. Cependant, dans certains cas, l'étude de la fonction de taux J est difficile. Le lemme ci-dessous propose une alternative, en utilisant la notion de tension exponentielle, qui est facile à obtenir avec nos hypothèses.

Lemme 1.2.5. *Si Y_t satisfait un Principe de Grandes Déviations faible avec une fonction de taux I et est exponentiellement tendue, i.e. pour tout $\alpha > 0$, il existe un compact K_α tel que*

$$\limsup_{t\to+\infty} \frac{1}{t} \log \mathbb{P}\left(Y_t \notin K_\alpha^c\right) < -\alpha,$$

alors Y_t satisfait un Principe de Grandes Déviations complet et I est une bonne fonction de taux.

Ce Lemme est une conséquence du Lemme 1.2.18 dans [25]. Le Théorème 1.2.4 et ce dernier Lemme 1.2.5 seront utilisés à la fin de la preuve, pour prouver le Théorème principal 1.2.7.

Résultats de Grandes Déviations

Nous faisons les hypothèses suivantes :

Hypothèse 1.2.6. *i)* $\exists \theta_0 \in (0, +\infty]$ tel que $\mathbb{E}[\mathrm{e}^{\theta\tau}] < \infty$ pour tout $\theta < \theta_0$,
ii) $\exists \beta_0 \in (0, +\infty]$ tel que $\mathbb{E}[\mathrm{e}^{\beta|W|}] < \infty$, pour tout $\beta < \beta_0$.

Nous introduisons les transformées de Cramer classiques, pour $(a, b) \in \mathbb{R}^2$,

$$\Lambda^*(a,b) = \sup_{x,y} \{ax + by - \ln \mathbb{E}(e^{x\tau + yW})\} \tag{1.2.4}$$

$$\Lambda_n^*(a,b) = \sup_{x,y} \left\{ax + by - \ln \mathbb{E}\left(e^{x\tau + yW^n}\right)\right\}, \tag{1.2.5}$$

où W^n est une réduction bien choisie de W. Nous introduisons finalement les fonctions de taux

$$J^n(m) = \inf_{\beta>0} \beta\, \Lambda_n^*\left(\frac{1}{\beta}, \frac{m}{\beta}\right) \quad , \quad J(m) = \inf_{\beta>0} \beta\, \Lambda^*\left(\frac{1}{\beta}, \frac{m}{\beta}\right), \tag{1.2.6}$$

et

$$\tilde{J} = \sup_{\delta>0} \liminf_{n\to\infty} \inf_{|m-z|<\delta} J^n(z)\,. \tag{1.2.7}$$

On peut ensuite établir

Théorème 1.2.7. *Supposons que les Hypothèses 1.2.6 soient satisfaites. Soit J définie par (1.2.6) et $\tilde{J}$ par (1.2.7). On écrit $m = \mathbb{E}(W)/\mathbb{E}(\tau)$.*

— *Si $\beta_0 = +\infty$ (en particulier, si W est borné) alors Z_t/t satisfait un Principe de Grandes Déviations complet avec $\tilde{J}$ pour bonne fonction de taux. Nous avons également les inégalités suivantes*

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{Z_t}{t} \geq m + a\right) \leq - \inf_{z\geq m+a} J(z), \tag{1.2.8}$$

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{Z_t}{t} \leq m - a\right) \leq - \inf_{z\leq m-a} J(z). \tag{1.2.9}$$

— *Si $\beta_0 < +\infty$, nous avons pour tout $a > 0$ et tout $\kappa \in (0,1)$*

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{Z_t}{t} \geq m + a\right) \leq -\min\left[\inf_{z\geq m+\kappa a} J(z)\ ,\ (1-\kappa)\beta_0 a/2\right], \tag{1.2.10}$$

et

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{Z_t}{t} \leq m - a\right) \leq -\min\left[\inf_{z\geq m-\kappa a} J(z)\ ,\ (1-\kappa)\beta_0 a/2\right]. \tag{1.2.11}$$

Remarque 1.2.8. Les inégalités précédentes (1.2.8) et (1.2.9) restent en fait vraies avec la fonction $\tilde{J}$ puisque $J \leq \tilde{J}$. Néanmoins, comme J est beaucoup plus facile à obtenir, nous préférons écrire cette version.

Remarque 1.2.9. En fait, nous considérons ici que $r_t = 0$ dans la définition de Z_t. Cette hypothèse peut être relâchée si r_t/t tend vers 0 assez vite, comme cela sera le cas pour les processus de Hawkes. Par exemple, si pour tout $\delta > 0$

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{|r_t|}{t} > \delta\right) = -\infty,$$

alors Z_t/t et $(Z_t - r_t)/t$ sont dits exponentiellement équivalents et satisfont le même principe de grande déviation (si l'un des deux en admet un, l'autre en admet également un) et ont la même fonction de taux. C'est cette situation qui s'applique pour les processus de Hawkes, sous réserve d'hypothèse supplémentaire.

Stratégie de preuve

La preuve de ce théorème est divisée en plusieurs étapes. Tout d'abord, nous commençons par réduire le problème aux W bornés, en remplaçant W par W^n, puis aux W à valeurs dans un ensemble fini. Sous de bonnes hypothèses, ces différents processus cumulatifs sont des approximations exponentiellement bonnes, et le Théorème 1.2.4 permettra de conclure. Il nous suffit alors de nous concentrer sur le cas des W à valeurs dans un ensemble fini.

En suivant la démarche de [52], nous introduisons une mesure empirique μ_t^n associée au processus et nous prouvons le Principe de Grandes Déviations pour cette mesure :

Théorème 1.2.10. *Soit μ_t^n la mesure empirique d'un processus cumulatif satisfaisant les Hypothèses 1.2.6, et tel que W^n prenne ses valeurs dans $\{1, ..., n\}$. Alors la famille $(\mathbb{P}_t^n)_{t\geq 0}$ des distributions de μ_t^n satisfait un principe de grande déviation avec une bonne fonction de taux I^n quand $t \to \infty$ avec une vitesse t, où I^n n'est pas précisée ici, mais est définie dans le corps du manuscrit, dans* (2.4.5).

Pour prouver ce Théorème, nous étudions la fonction de taux de la mesure empirique. Il s'agit d'une étude assez technique. Nous adaptons les arguments de [52] en utilisant des conditionnements par rapport à chaque valeur de W. Le fait que W soit à valeurs dans un ensemble fini nous permet de conditionner sans difficulté

et nous permet d'utiliser des arguments de compacité puisque $\mathcal{M}_1(\{1, ..., n\})$ est compact.
De nombreux arguments sont proches de [52], mais nous devions écrire plus en détail certaines preuves. Certains points topologiques devaient également être clarifiés. Pour l'article qui a été soumis et que nous reprenons ici, nous avons fait le choix de faire référence aux résultats correspondants dans [52] seulement lorsqu'ils pouvaient être reproduits ligne à ligne dans notre cas. Nous avons ajouté en annexe les preuves venant de [52].
Le reste de la preuve de ce Théorème est similaire au travail effectué par [52].

Le principe de contraction pour les Principes de Grandes Déviations nous permet finalement de transférer ce principe de la mesure empirique au processus cumulatif approché (de cumulants bornés, et à valeurs dans un ensemble fini). Pour ce faire, nous avons dû travailler sur la fonction de taux associée afin d'en extraire une écriture la plus claire possible.
Il est à noter que, si nous avons réussi à prouver un principe dit complet pour les processus cumulatifs approchés (de cumulants bornés, et à valeurs dans un ensemble fini), nous n'obtenons qu'un principe dit faible dans le cadre général. Nous nous restreignons avec des hypothèses sur la loi de W pour assurer un principe complet.

1.2.1.3 Chapitre 3 : Propriétés asymptotiques pour les processus de Hawkes

Définitions
Nous considérons un espace de probabilité muni d'une filtration appropriée $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ qui satisfait les hypothèses habituelles.

Définition 1.2.11. Soit $\lambda > 0$ et $h : (0, +\infty) \to \mathbb{R}$ une fonction mesurable signée. Soit N^0 un processus ponctuel, localement fini, sur $(-\infty, 0]$, qui suit la loi $\mathbf{m}$.
Le processus ponctuel N^h sur $\mathbb{R}$ est un processus de Hawkes sur $(0, +\infty)$, de condition initiale N^0 et de mesure de reproduction $\mu(dt) = h(t)dt$ si :
- $N^h |_{(-\infty,0]} = N^0$,
- la mesure d'intensité $N^h |_{(0,+\infty)}$ conditionnellement à $(\mathcal{F}_t)_{t\geq 0}$ est absolument continue par rapport à la mesure de Lebesgue et a pour densité :

$$\Lambda^h : t \in (0, +\infty) \mapsto \left(\lambda + \int_{(-\infty,t)} h(t-u)N^h(du)\right)^+. \tag{1.2.12}$$

où $x^+ = \max(x, 0)$.

On nomme h *fonction de reproduction* du processus, ou *noyau de reproduction.*

Remarque 1.2.12. Il est possible de définir un processus de Hawkes plus général, en choisissant

$$\Lambda^h(t) = f\left(\int_{(-\infty,t)} h(t-u)N^h(du)\right),$$

avec f une fonction positive appelée *fonction de taux de saut*. Dans ce travail, nous traitons seulement le cas $f : y \mapsto \max(0, \lambda + y)$.
On remarque que si h est une fonction positive, on peut remplacer $f : y \mapsto \max(0, \lambda + y)$ par une fonction affine $y \mapsto \lambda + y$. On qualifie ces processus de Hawkes de *linéaire*. Ce sont les processus de Hawkes historiques, définis par Oakes et Hawkes [44], qui ont été particulièrement étudiés. Nous utiliserons dans la suite certains résultats obtenus pour ceux-ci pour les étendre aux processus de Hawkes avec $f : y \mapsto \max(0, \lambda + y)$.

La proposition suivante donne une représentation explicite d'un processus de Hawkes comme une solution d'une EDS dirigée par un mesure aléatoire de Poisson sur $\mathbb{R}^2$, et établit une propriété de couplage très importante.

Proposition 1.2.13 (Proposition 2.1 de [22]). *Soit Q une mesure aléatoire de Poisson de dimension deux, sur $(0, +\infty) \times (0, +\infty)$, d'intensité unitaire $dxdy$ et adaptée à $(\mathcal{F}_t)_{t \geq 0}$. Nous considérons le système*

$$\begin{cases} N^h = N^0 + \int_{(0,+\infty)\times(0,+\infty)} \delta_u \mathbb{1}_{\theta \leq \Lambda^h(u)} Q(du, d\theta) \\ \Lambda^h(u) = \left(\lambda + \int_{(-\infty,u)} h(u-s) N^h(ds)\right)^+, \ u > 0, \end{cases} \tag{1.2.13}$$

où $\lambda > 0$ est un taux d'immigration, $h : (0, +\infty) \to \mathbb{R}$ est une fonction mesurable signée et N^0 est une condition initiale de loi $\mathbf{m}$ sur $(-\infty, 0]$.
Nous considérons le système similaire pour N^{h^+}, dans lequel h est remplacé par $h^+(.) = max(h(.), 0)$. Nous supposons que $\|h^+\|_1 :=\| h^+ \|_{L^1(du)} < 1$ et que la distribution $\mathbf{m}$ satisfait :

$$\forall t > 0, \int_0^t \mathbb{E}_{\mathbf{m}} \left(\int_{(-\infty,0]} h^+(u-s) N^0(ds) \right) du < +\infty. \tag{1.2.14}$$

Alors :
- *Il existe une solution forte, avec unicité trajectorielle N^h de l'équation* (1.2.13) *et cette solution est un processus de Hawkes.*
- *Cette propriété est également vraie pour N^{h^+}. Par ailleurs, $N^h \leq N^{h^+}$ au sens des mesures, ce qui signifie que pour tous $0 \leq s \leq t < +\infty$, $N^h([s,t]) \leq N^{h^+}([s,t])$.*

Les processus de Hawkes caractérisés par une fonction h positive ont été particulièrement étudiées. Nous avons choisi de considérer des processus définis avec une fonction de reproduction h signée, dans le but de représenter une potentielle inhibition du processus.

Hypothèse 1.2.14. Dans ce travail, nous ferons les hypothèses suivantes :

i) $h : (0, +\infty) \to \mathbb{R}$ est une fonction mesurable signée à support compact. Soit $L(h)$ le supremum du support de h : $L(h) := \sup\{t > 0, |h(t)| > 0\} < \infty$.

ii)
$$\|h^+\|_1 := \int_0^{+\infty} h^+(u)\, du < 1,$$
où $h^+(x) = max(h(x), 0)$.

iii) $\lambda > 0$,
iv) la condition initiale sur $]-\infty, 0[$ ne contient aucun saut i.e. $\mathbf{m} = \delta_\emptyset$.

Nous nous intéressons au comportement asymptotique du nombre de sauts du processus N^h sur l'intervalle $[0, t]$ que nous notons :

$$N_t^h = N^h([0,t]), \quad \forall t \geq 0$$

En particulier, puisque des résultats sont déjà connus lorsque h est positive, nous souhaitons quantifier la perte de points dûe à l'inhibition.

La Proposition 1.2.13 nous permet d'établir une majoration de N_t^h par $N_t^{h^+}$. Nous souhaitions également obtenir une minoration, ce que nous permet la Proposition suivante

Proposition 1.2.15 (Minoration d'un processus de Hawkes)**.**
Soit h une fonction satisfaisant les Hypothèses 1.2.14. Soit $\lambda > 0$, et définissons $g = -\lambda \mathbb{1}_{[0,L(h)]}$.
On peut trouver un couplage de deux processus de Hawkes N^h et N^g, associés respectivement aux fonctions de reproduction h et g, et d'intensité de base λ, tel que pour tout $t \geq 0$:

$$N_t^h \geq N_t^g \quad p.s.$$

Il faut noter que ce résultat de comparaison est plus faible que la majoration via h^+. En effet, nous n'avons pas $N^h([s,t]) \geq N^g([s,t])$ pour tout s, mais seulement pour $s = 0$.
Ce résultat motive l'étude détaillée de N_t^g pour $g = -\lambda \mathbb{1}_{[0,A]}$, avec $A > 0$, qu'on appelle le cas d'annulation de l'intensité. Cette étude a été faite en particulier dans le Chapitre 3, dans la remarque 3.2.8, et permet d'obtenir des bornes pour certaines propositions.

Les processus de Hawkes sont des processus cumulatifs
Soit N^h un processus de Hawkes, qui suit les Hypothèses 1.2.14, avec pour condition initiale $N^0 = \emptyset$. Ce processus de Hawkes peut être en fait vu comme un processus cumulatif.

On écrit $U_1, U_2, U_3, \ldots$ ses sauts successifs.
Nous introduisons les temps de renouvellement du processus, qui permettent de diviser la demi-droite temporelle $\mathbb{R}^+$ en fenêtres temporelles (ou intervalles) du processus, qui sont indépendantes et identiquement distribuées. Leurs durées sont données par $\tau_1, \tau_2, \ldots$.

Commençons par définir le temps d'arrêt

$$\tau_1 = \inf\{t > U^1, N^h((t - L(h), t]) = 0\},$$

qui est le premier instant après U^1 tel qu'il n'y a pas eu de saut pendant une durée $L(h)$. Nous posons également

$$S_0 = 0 \quad \text{et} \quad S_1 = \tau_1.$$

Définissons maintenant

$$W_1 = N^h([U^1, S_1]) = N^h([0, S_1]),$$

le nombre de sauts du processus dans la première fenêtre temporelle. Nous renumérotons les instants de sauts dans cette première fenêtre temporelle :

$$U_j^1 = U_j, \quad \forall j \in \{1, \ldots, W_1\}.$$

Nous verrons par la suite que τ_1 et W_1 sont finis presque sûrement. Par récursivité, soit $i \in \mathbb{N}^*$ tel que $(\tau_1, W_1), ...(\tau_i, W_i)$ sont bien définis (et finis p.s.). Soit $S_i = \sum_{k=1}^{i} \tau_k$. Nous définissons

$$U_1^{i+1} = U_{W_1+\ldots+W_i+1},$$

et

$$\tau_{i+1} = \inf\{t > U_1^{i+1}, N^h((t - L(h), t]) = 0\} - S_i, \tag{1.2.15}$$

On peut remarquer qu'il y a au moins un saut dans $[S_i, S_i + \tau_{i+1}]$. Nous définissons le nombre de sauts dans la $(i + 1)$ème fenêtre par

$$W_{i+1} = N^h([U_1^{i+1}, S_i + \tau_{i+1}) = N^h([S_i, S_i + \tau_{i+1}]), \tag{1.2.16}$$

et nous renumérotons les sauts de cette fenêtre par :

$$U_j^{i+1} = U_{W_1+\ldots+W_i+j}, \quad \forall j \in \{1, ..., W_{i+1}\}.$$

La Proposition suivante nous permet d'apporter des propriétés importantes sur la loi jointe des $(\tau_i, W_i)_i$. Il est néanmoins difficile d'apporter des informations plus explicites sans se restreindre à des cas spécifiques.

Proposition 1.2.16. *Sous les Hypothèses 1.2.14, on a :*

i) les $(\tau_i, W_i)_i$ sont des variables aléatoires i.i.d.,

ii) pour $i \in \mathbb{N}^*$, les durées $(U_1^i - S_{i-1})$ sont des variables aléatoires i.i.d. de distribution exponentielle $\mathcal{E}(\lambda)$. Cela signifie que la durée entre le début d'une fenêtre et le premier saut de cette fenêtre suit une loi exponentielle.

On peut écrire

$$N_t^h = \sum_{i=1}^{\infty} \mathbb{1}_{U_i \le t} = \sum_{i=1}^{\infty} \sum_{j=1}^{W_i} \mathbb{1}_{U_i^j \le t}\,. \tag{1.2.17}$$

Introduisons le processus de renouvellement associé aux S_i

$$M_t^h := \sum_{i=1}^{\infty} \mathbb{1}_{S_i \le t}\,. \tag{1.2.18}$$

Puisque $S_i = \sum_{k=1}^{i} \tau_k$, on peut définir

$$\hat{N}_t^h := \sum_{i=1}^{\infty} W_i \mathbb{1}_{S_i \le t} = \sum_{i=1}^{M_t^h} W_i. \tag{1.2.19}$$

Pour tout $t \in \mathbb{R}^+$, la fenêtre à l'instant t est la $M_t^h + 1$-ème. $\hat{N}_t^h$ ne contient que les sauts du processus jusqu'à la M_t^h-ème fenêtre (incluse) tandis que N_t^h peut contenir plus de sauts. En particulier,

$$\hat{N}_t^h \leq N_t^h \leq \hat{N}_t^h + W_{M_t^h+1} \quad \text{p.s.} \tag{1.2.20}$$

On a alors

$$N_t^h = \sum_{i=1}^{M_t^h} W_i + R_t^h$$

pour un processus de renouvellement M_t^h, avec un terme restant $R_t^h \leq W_{M_t^h+1}$, et pour des W_i i.i.d. Il s'agit donc d'un processus cumulatif.

Résultats asymptotiques

Nous pouvons maintenant énoncer les résultats principaux de ce chapitre. La clé est d'obtenir suffisamment de moments pour les (τ_i, W_i). La première propriété règle ce problème.

Proposition 1.2.17. *Soit h une fonction signée satisfaisant les Hypothèses 1.2.14. On considère le processus de Hawkes N^h et les couples de variables aléatoires (τ_i, W_i) i.i.d. définis dans (1.2.15)-(1.2.16). Alors*

i) Pour $\alpha < \alpha_0 := \min\left(\lambda\,,\ \frac{||h^+||_1 - \ln(||h^+||_1) - 1}{L(h)}\right)$ on a $\mathbb{E}(e^{\alpha\tau_1}) < +\infty$.

ii) Il existe $\theta_0 > 0$ tel que pour tout $\theta < \theta_0$, $\mathbb{E}(e^{\theta W_1}) < +\infty$.

En particulier τ_1 et W_1 ont des moments polynomiaux de tout ordre.

Proposition 1.2.18. *Si $h \leq 0$, on peut choisir $\theta_0 < -\ln\left(1 - e^{-\lambda L(h)}\right)$ dans la proposition précédente 1.2.17.*

Grâce à cette première propriété et à la structure de renouvellement des processus de Hawkes, on peut en déduire les résultats asymptotiques suivants :

Théorème 1.2.19 (**Loi des Grands Nombres).** *Soit h une fonction signée qui satisfait les Hypothèses 1.2.14 et N^h le processsus de Hawkes donné par (1.2.13). Alors*

$$\frac{N_t^h}{t} \xrightarrow[t\to\infty]{p.s.} \frac{\mathbb{E}[W_1]}{\mathbb{E}(\tau_1)}\,.$$

Grâce à nos résultats de comparaison et à l'étude précise du cas d'annulation d'intensité (N_t^g pour $g = -\lambda \mathbb{1}_{[0,A]}$) obtenus dans la Remarque 3.2.8 dans le Chapitre 3, on a

$$\frac{\lambda}{1 + \lambda L(h)} \leq \frac{\mathbb{E}[W_1]}{\mathbb{E}(\tau_1)} \leq \frac{\lambda}{1 - ||h^+||_1}\,.$$

Cette méthode nous fournit également un Théorème Central Limite.

Théorème 1.2.20 (Théorème Central Limite). *Soit h une fonction signée qui satisfait les Hypothèses 1.2.14 et N^h le processsus de Hawkes donné par (1.2.13). Alors*

$$\sqrt{t}\left(\frac{N_t^h}{t} - \frac{\mathbb{E}[W_1]}{\mathbb{E}(\tau_1)}\right) \underset{t\to\infty}{\Longrightarrow} \mathcal{N}(0,\sigma^2)$$

avec

$$\sigma^2 = \frac{Var\left(W_1 - \tau_1 \frac{\mathbb{E}[W_1]}{\mathbb{E}(\tau_1)}\right)}{\mathbb{E}(\tau_1)}.$$

On obtient finalement des résultats de grandes déviations, grâce au travail effectué sur les processus cumulatifs dans le chapitre précédent. Nous rappelons et adaptons les notations dans ce contexte.

Définition 1.2.21. Nous définissons la transformée de Cramer pour $(a,b) \in \mathbb{R}^2$,

$$\Lambda^*(a,b) = \sup_{x,y}\left\{ax + by - \ln\left(\mathbb{E}\left[e^{x\tau_1 + yW_1}\right]\right)\right\}.$$

Nous définissons également, pour $z \in \mathbb{R}^+$,

$$J(z) = \inf_{\beta>0}\left(\beta\,\Lambda^*\left(\frac{1}{\beta}, \frac{z}{\beta}\right)\right).$$

De manière similaire, Λ_n^* et J_n sont définis en remplaçant W_1 par $\min(W_1, n)$ (Ici, les $\min(W_1,n)$ sont déjà à valeurs dans un ensemble fini). Finalement, on définit

$$\tilde{J}(z) = \sup_{\delta>0}\liminf_{n\to\infty}\inf_{|y-z|<\delta} J_n(y).$$

Grâce à la Proposition 1.2.17 on peut appliquer le Théorème 1.2.7 des processus cumulatifs qui nous indique que le processus approché $\hat{N}_t^h/t$ satisfait les inégalités de déviations asymptotiques

Théorème 1.2.22. *Rappelons que θ_0 est défini dans la Proposition 1.2.17 (ii).*

— *Si $\theta_0 = +\infty$, la loi de la famille $\hat{N}_t^h/t$ satisfait un Principe de Grandes Déviations avec pour fonction de taux $\tilde{J}$, i.e*

  1. *pour tout fermé $\mathcal{C} \in \mathbb{R}$,*

$$\limsup_{t\to\infty}\frac{1}{t}\ln\mathbb{P}\left(\hat{N}_t^h/t \in \mathcal{C}\right) \leq -\inf_{m\in\mathcal{C}}\tilde{J}(m)$$

  2. *pour tout ouvert $\mathcal{O} \in \mathbb{R}$,*

$$\liminf_{t\to\infty}\frac{1}{t}\ln\mathbb{P}\left(\hat{N}_t^h/t \in \mathcal{O}\right) \geq -\inf_{m\in\mathcal{O}}\tilde{J}(m),$$

— *Si $\theta_0 < +\infty$, en notant $m = \mathbb{E}(W_1)/\mathbb{E}(\tau_1)$, on a pour tout $a > 0$ et tout $\kappa \in (0,1)$*

$$\limsup_{t\to+\infty}\frac{1}{t}\ln\mathbb{P}\left(\frac{\hat{N}_t^h}{t} \geq m + a\right) \leq -\min\left[\inf_{z\geq m+\kappa a} J(z)\,,\ (1-\kappa)\theta_0 a\right],$$

et

$$\limsup_{t\to+\infty}\frac{1}{t}\ln\mathbb{P}\left(\frac{\hat{N}_t^h}{t}\leq m-a\right)\leq-\min\left[\inf_{z\geq m-\kappa a}J(z)\ ,\ (1-\kappa)\theta_0 a\right].$$

Les dernières inégalités sont obtenues en utilisant le fait que $J\leq\tilde{J}$.

Nous prouvons finalement le corollaire suivant

Corollaire 1.2.23. *On rappelle que θ_0 est défini dans la Proposition 1.2.17 (ii).*

(1) Si $\theta_0=+\infty$, N_t^h/t satisfait le même Principe de Grandes Déviations que $\hat{N}_t^h/t$.

(2) Si $\theta_0<+\infty$, on a pour tout $a>0$ et $\kappa\in(0,1)$

$$\limsup_{t\to\infty}\frac{1}{t}\ \ln\mathbb{P}\left(\frac{N_t^h}{t}\ \geq\ m+a\right)\leq\ -\ \min\left[\inf_{z-m\geq\kappa a}\ J(z)\,,\ \frac{1-\kappa}{2}\theta_0 a\right], \tag{1.2.21}$$

De façon similaire

$$\limsup_{t\to\infty}\frac{1}{t}\ \ln\mathbb{P}\left(\frac{N_t^h}{t}\ \leq\ m-a\right)\leq\ -\ \min\left[\inf_{m-z\leq\kappa a}\ J(z)\,,\ \frac{(1-\kappa)}{2}\theta_0 a\right]. \tag{1.2.22}$$

avec $\kappa\in(0,1)$.

Stratégie de preuve

L'idée initiale pour travailler sur les propriétés asymptotiques de ce processus a été d'utiliser la structure sous-jacente de renouvellement montrée par [22]. C'est ainsi que nous avons pu montrer que les processus de Hawkes, lorsqu'ils sont associés à une fonction h à support compact, pouvaient être considérés comme des processus cumulatifs.
Pour étudier les moments de τ et de W, il suffisait de rechercher les moments de variables τ^+ et W^+ qui apparaissent pour le processus N^{h^+}, et qui majorent respectivement τ et W. En ce qui concerne les moments exponentiels de τ^+, nous avons pu comparer cette variable aléatoire à une quantité connue dans la théorie des files d'attente : il s'agit de la durée d'occupation de la file d'attente. Cette comparaison nous a permis d'utiliser des propriétés connues sur sa loi pour obtenir cette proposition.
Les moments de W^+ ont été obtenus grâce à la forme $W^+=N^{h^+}([0,\tau^+])$, et à l'utilisation des propriétés obtenues dans [8] dans le cadre des Hawkes linéaires.

Une fois cette étude sur τ et W menée, nous avons pu appliquer les théorèmes bien connus pour les processus cumulatifs à une approximation $\hat{N}_t^h$ du processus de Hawkes. Il a ensuite fallu être vigilants sur la loi de la quantité $r_t=N_t^h-\hat{N}_t^h$. C'est cette quantité qui nous fait légèrement perdre sur la majoration dans les inégalités de grandes déviations (au lieu d'avoir une majoration par $-(1-\kappa)\theta_0 a$, on a une majoration en $-(1-\kappa)\theta_0 a/2$ dans (1.2.21) et (1.2.22)).

1.2.1.4 Perspectives

Finalement, ces deux chapitres exhibent une vision nouvelle des processus de Hawkes, sous les hypothèses étudiées, comme processus cumulatifs. Ce point de vue permet non seulement d'établir une loi des grands nombres et un théorème central limite pour les processus de Hawkes, mais également un principe de grandes déviations. Nous démontrons ce principe pour les processus cumulatifs en démontrant un principe sur les mesures empiriques, suivant les travaux de [52].

L'originalité de cette étude réside dans le lien entre processus cumulatif et processus de Hawkes. Néanmoins, ce lien repose sur les hypothèses que nous imposons aux processus de Hawkes, à savoir avoir une fonction de reproduction h à support compact. Si cette hypothèse ne pose en général pas de problème pour la modélisation en première approche (considérer qu'un processus n'a qu'une dépendance limitée dans le temps, en acceptant des temps très longs, n'est pas très restrictif), il pourrait être intéressant d'étudier d'autres hypothèses moins fortes. Ainsi, Carl Graham [38] a étendu le travail de [22] sur des propriétés regénératives de processus de Hawkes. Par ailleurs, la pré-publication [69] expose également une structure de renouvellement pour des processus de Hawkes dans un cadre général. Il serait donc intéressant de déterminer si ces cadres permettent la comparaison à un processus cumulatif associé.
De plus, l'une des difficultés de la mise en œuvre des théorèmes démontrés est l'absence de lien simple entre la fonction de reproduction h et la loi du couple (τ, W). Sans forcément déterminer exactement la loi du couple à partir de h, nous pourrions explorer les liens entre des hypothèses sur h et des conséquences sur la loi de (τ, W).

Le travail que nous avons effectué ici était dans le cadre d'une fonction de taux de saut $f : y \mapsto (y + \lambda)^+$ (voir la Remarque 1.2.12). Cela pourrait être étendu à des fonctions de taux plus générales, à condition de conserver une structure de renouvellement. A ma connaissance, cela n'a pas encore été vérifié - mis à part les cadres plus généraux de Graham et Raad exposés ci-dessus - et il serait intéressant de s'y plonger.

Ce travail sur les propriétés asymptotiques du processus N_t^h/t est un travail théorique, finalement assez loin des applications. Néanmoins, pour la modélisation de neurones, cette quantité est en fait un taux de décharge au cours du temps. Il pourrait donc être pertinent d'étudier des statistiques pour le processus. Ainsi, plusieurs études statistiques travaillent sur des estimations d'intensité de processus [72] ou des tests [30] permettant par exemple de distinguer des processus de Poisson d'autres types de processus. Nous pourrions donc commencer par travailler sur des tests pour distinguer un processus de Hawkes dont l'intensité s'annule (pour lequel nous avons des calculs explicites) d'un processus de Poisson homogène ; puis estimer les paramètres d'un tel processus de Hawkes.
De plus, nous souhaitons également obtenir des propriétés de concentration (c'est-à-

dire de déviation en temps fini) pour les processus cumulatifs. De telles propriétés, appliquées ensuite aux processus de Hawkes, pourraient nous permettre d'obtenir des estimateurs avec une précision connue.

Enfin, de nombreuses questions restent en suspend pour les processus cumulatifs en eux-mêmes. Même s'il s'agit de processus "naturels", peu de leurs propriétés ont été étudiées dans un cadre général. Nous pouvons nous intéresser à la loi des trajectoires $Z_{[0,t]}$, étendre les résultats de grandes déviations à un cadre plus général ou étudier plus spécifiquement la fonction de taux.

1.2.2 Processus de FitzHugh-Nagumo - Chapitre 4

1.2.2.1 Définition et propriétés

Le modèle de FitzHugh-Nagumo déterministe, pour un unique neurone (ou une particule), se décrit de la façon suivante : on considère $Z_t \in \mathbb{R}^2$ l'état du neurone. Z_t peut s'écrire (X_t, C_t) et évolue selon le système

$$\begin{cases} dX_t = (X_t - (X_t)^3 - C_t - \alpha)dt \\ dC_t = (\gamma X_t - C_t + \beta)dt, \end{cases}$$

où X est le potentiel de membrane et C est une variable dite de récupération, appelée variable d'adaptation. Les paramètres γ et β sont des constantes positives qui déterminent la durée d'une excitation et la position du point d'équilibre de ce système. $\alpha \in \mathbb{R}$ est la magnitude du courant de stimulus (un courant d'entrée dans le système, pouvant être négatif, nul ou positif).
C n'est pas une quantité physiologique, mais est utilisée pour créer une phase de récupération, ou de remise au repos du potentiel, après la décharge. En fait, elle est mathématiquement utilisée pour permettre à X de modéliser au mieux le vrai comportement du potentiel. C a une dynamique linéaire, et croît lorsque X croît. Ensuite, lorsqu'elle est suffisamment importante, elle génère une force de rappel sur X pour forcer un retour à des valeurs plus faibles.

Nous étudions ce modèle dans un cadre stochastique. Il est possible de considérer du bruit sur chacune de ces équations. Un bruit sur la dynamique de la variable X modélise un courant présynaptique bruité, c'est-à-dire un courant bruité qui est reçu par le neurone. Un bruit sur la dynamique de C représente un bruit dans le comportement chimique du système. Nous avons fait le choix de considérer du bruit sur les deux équations. Néanmoins, notre étude permet de choisir un écart-type σ_X (pour le bruit sur la dynamique de dX_t) ou un écart-type σ_C (pour la dynamique de C_t) nul, et donc de ne considérer qu'un seul bruit. Nous nous sommes concentrés sur le cas d'un bruit Brownien, comme c'est le cas en général dans la littérature [81, 53, 82].

Nous nous intéressons à un ensemble de N neurones en interaction. Pour simplifier cette étude, nous considérons que les paramètres du modèle sont les mêmes

pour tous les neurones et qu'il n'y a pas de variabilité individuelle (α, β, γ, σ_C et σ_X ne dépendent pas de l'indice i). Le modèle est décrit ci-dessous (1.2.23).
Afin de modéliser les interactions, nous supposons que tous les neurones sont connectés les uns aux autres, et que les apports de chacun des neurones sont les mêmes. Ainsi, pour éviter une explosion du modèle, nous supposons les apports de chaque neurone de l'ordre de $1/N$: nous travaillons donc dans un cadre *champ-moyen*.
Ces apports sont décrits par deux fonctions K_X et K_C, qui sont ajoutés sur la dynamique de chaque coordonnée. Nous nous concentrons sur des interactions chimiques. Dans ce cas, la fonction d'interaction est appliquée à la différence entre l'état du neurone qui envoie de l'information et l'état de celui qui le reçoit, i.e. l'apport du neurone j sur le neurone i est donné par $K_X(Z^i - Z^j)$ et $K_C(Z^i - Z^j)$.

Finalement, le modèle de FitzHugh-Nagumo que nous étudions est le suivant : on considère un réseau de $N \in \mathbb{N}^*$ neurones. Pour $1 \leq i \leq N$, l'état du i-ème neurone à l'instant t est caractérisé par $Z_t^{i,N} = (X_t^{i,N}, C_t^{i,N}) \in \mathbb{R}^2$. Les Z_t^i suivent le système suivant :

$$\begin{cases} dX_t^{i,N} = (X_t^{i,N} - (X_t^{i,N})^3 - C_t^{i,N} - \alpha)dt + \frac{1}{N}\sum_{j=1}^N K_X(Z_t^{i,N} - Z_t^{j,N}) + \sigma_X dB_t^{i,X} \\ dC_t^{i,N} = (\gamma X_t^{i,N} - C_t^{i,N} + \beta)dt + \frac{1}{N}\sum_{j=1}^N K_C(Z_t^{i,N} - Z_t^{j,N}) + \sigma_C dB_t^{i,C}, \end{cases} \tag{1.2.23}$$

A i fixé, $Z^{i,N}$ représente l'état du i-ème neurone dans un réseau de N neurones. On suppose $(B_t^{i,X})_t$ et $(B_t^{i,C})_t$ des mouvements browniens indépendants.

La propagation du chaos
Pour décrire le comportement de ce réseau quand le nombre de neurones tend vers l'infini, une notion importante est la propagation du chaos, que nous redéfinissons rapidement ici.

Nous travaillons dans le contexte d'une loi symétrique, c'est-à-dire que les particules sont interchangeables.

Définition 1.2.24. Un N-uplet de variables aléatoires $Z^N = (Z^{1,N}, \dots, Z^{N,N})$ suit une loi dite *symétrique* si cette loi est invariante par permutation du N-uplet. En d'autres termes, pour toute permutation $\sigma : \{1, \dots, N\} \to \{1, \dots, N\}$, la loi de $(Z^{\sigma(1),N}, \dots, Z^{\sigma(N),N})$ est la même que la loi de Z^N.

En 1956, Kac [50] définit la notion de chaos, en l'appelant "propriété de Boltzmann" - en référence aux travaux de ce dernier sur les collisions d'atomes dans les gazs parfaits. Cette notion a été ensuite développée par Sznitman [77]. Nous la donnons ici dans $\mathbb{R}^d$, mais cette définition peut se généraliser.

Définition 1.2.25. Pour tout $N \in \mathbb{N}$, soit $Z^N = (Z^{1,N}, \dots, Z^{N,N})$ une suite de variables aléatoires de loi symétrique sur $(\mathbb{R}^d)^N$. On note ν_N la loi de Z^N. Soit ν une mesure de probabilité sur $\mathbb{R}^d$.

On dit que la suite $(\nu_N)_N$ est ν-chaotique si pour tout entier k, et pour toutes fonctions tests continues bornées $\varphi_1, \dots, \varphi_k$, on a

$$\int \varphi_1(x_1) \times \cdots \times \varphi_k(x_k) \nu_N(dx_1, \dots, dx_k) \underset{N\to\infty}{\longrightarrow} \left(\int \varphi_1(x)\nu(dx)\right) \times \cdots \times \left(\int \varphi_k(x)\nu(dx)\right).$$

Cela signifie qu'à k fixé, k particules dans un ensemble de N particules sont asymptotiquement i.i.d. quand N tend vers l'infini.

La notion de propagation du chaos en découle assez naturellement : si on considère un système qui part de conditions initiales chaotiques, on souhaite vérifier que cette propriété "se propage" au cours du temps.

Définition 1.2.26. Soit un système de particules qui évolue selon une certaine évolution. On suppose que la condition initiale de ce système de N particules suit la loi $\nu_{0,N}$. On note $\nu_{t,N}$ la loi de ce système au temps t.
On dit qu'il y a propagation du chaos, si lorsque $\nu_{0,N}$ est chaotique, alors pour tout $t > 0$, $\nu_{t,N}$ est chaotique.

Cela signifie qu'à t fixé, si le système de N particules est initialement chaotique, alors il le reste à l'instant t. Ainsi, à k et t fixés, les k particules parmi N, observées à l'instant t, sont asymptotiquement i.i.d. lorsque le nombre de particules N tend vers l'infini.

1.2.2.2 Principaux résultats

Pour décrire le comportement du modèle de FitzHugh-Nagumo défini en (1.2.23), nous le comparons au modèle suivant : on définit le processus $(\bar{Z}_t)_{t\geq 0} = (\bar{X}_t, \bar{C}_t)_{t\geq 0}$ comme un processus à valeurs dans $\mathbb{R}^2$ qui évolue suivant l'équation différentielle stochastique de McKean-Vlasov non linéaire suivante

$$\begin{cases} d\bar{X}_t = (\bar{X}_t - (\bar{X}_t)^3 - \bar{C}_t - \alpha)dt + K_X * \bar{\mu}_t(\bar{Z}_t)dt + \sigma_x d\bar{B}_t^X \\ d\bar{C}_t = (\gamma\bar{X}_t - \bar{C}_t + \beta)dt + K_C * \bar{\mu}_t(\bar{Z}_t)dt + \sigma_c d\bar{B}_t^C, \end{cases} \tag{1.2.24}$$

où $\bar{\mu}_t$ est la loi du processus $\bar{Z}_t$ au temps t et où $*$ décrit l'opération de convolution

$$K_X * \bar{\mu}_t(u) = \int K_X(u-v)\mu_t(dv).$$

D'une certaine manière, (1.2.24) peut être vue comme la limite de (1.2.23), puisque l'interaction dans (1.2.23) peut alors être vue comme une convolution de K_X ou K_C avec la mesure empirique $\mu_{t,\text{emp}} = \frac{1}{N}\sum_{i=1}^N \delta_{Z_t^i}$.

Nous obtenons tout d'abord un résultat d'existence - pour s'assurer qu'il existe bien une solution aux deux systèmes précédents - puis deux résultats de propagation du chaos. Ces résultats sont sous les hypothèses suivantes sur les noyaux d'interaction :

Hypothèse 1.2.27. Soit $L_{X,max}$ et $L_{C,max}$ deux constantes universelles.

$$\exists L_X \in [0, L_{X,max}), \forall z, z' \in \mathbb{R}^2 \quad |K_X(z) - K_X(z')| \leq L_X \|z - z'\|_1$$
$$\exists L_C \in [0, L_{C,max}), \forall z, z' \in \mathbb{R}^2 \quad |K_C(z) - K_C(z')| \leq L_C \|z - z'\|_1$$
$$K_X(0,0) = 0 \text{ and } K_C(0,0) = 0$$

Pour la propagation du chaos non uniforme en temps, cette hypothèse sur $L_{X,max}$ et $L_{C,max}$ peut être ignorée : dans ce cas, les valeurs choisies peuvent être $+\infty$, et l'important est que les deux fonctions d'interaction K_X et K_C soient Lipschitz.
Par contre, pour la propagation du chaos uniforme en temps, il sera nécessaire d'avoir des constantes de Lipschitz L_X et L_C assez petites. Cette condition sur $L_{X,max}$ et $L_{C,max}$ ne sera en fait pas la condition restrictive. Néanmoins, celle-ci permet de définir d'autres paramètres de manière indépendante de L_X et de L_C, paramètres qui permettent ensuite de définir des conditions sur L_X et L_C. En fait, la Remarque 4.2.7 de ce chapitre indique qu'on peut choisir pour ce théorème $L_{X,max} = 4$ et $L_{C,max} = \frac{1}{5}$.

On note $\mathcal{W}_1$ et $\mathcal{W}_2$ les distances de Wassertein L^1 et L^2 usuelles.

Tout d'abord, il est nécessaire de vérifier que les deux systèmes (1.2.23) et (1.2.24) ont des solutions bien définies :

Proposition 1.2.28 (Existence de solutions). *Soit K_X et K_C satisfaisant l'Hypothèse 1.2.27, avec $L_{X,max} = \infty$ et $L_{C,max} = \infty$.*
Il existe une unique solution forte pour le système (1.2.23) et une unique solution forte pour le système (1.2.24).

Nous pouvons maintenant énoncer le premier résultat de propagation du chaos :

Théorème 1.2.29 (Propagation du chaos non-uniforme en temps). *Soit K_X et K_C deux fonctions satisfaisant les Hypothèses 1.2.27, avec $L_{X,max} = \infty$ et $L_{C,max} = \infty$.*
Il existe deux constantes explicites C_1 et C_2 strictement positives telles que pour toute mesure de probabilité μ_0 sur $\mathbb{R}^2$ qui admet un second moment fini,

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq C_1 e^{C_2 t} \frac{k}{\sqrt{N}},$$

pour tout $k \in \mathbb{N}$, où $\mu_t^{k,N}$ est la distribution marginale, à l'instant t, des k premiers neurones $(Z_t^{1,N}, \ldots, Z_t^{k,N})$ d'un système de N neurones suivant le système (1.2.23), avec pour distribution initiale $(\mu_0)^{\otimes N}$, et où $\bar{\mu}_t$ est une solution de (1.2.24) avec pour distribution initiale μ_0.

Ce résultat, bien qu'à première vue plus faible que le résultat suivant, n'a pas de condition sur les coefficients de Lipschitz des fonctions d'interaction. De plus, sa preuve permet de comprendre la démarche pour le théorème principal.

Notre principal résultat est d'avoir réussi à enlever la dépendance en temps dans la borne supérieure précédente. Néanmoins, cette uniformité en temps s'accompagne d'hypothèses plus fortes sur les noyaux d'interaction.

Théorème 1.2.30 (Propagation du chaos uniforme en temps). *Soit $\tilde{\mathcal{C}^0} > 0$ et $\tilde{a} > 0$. Il existe un $\mathcal{C}_{K_X,K_C} > 0$ explicite tel que, pour tout K_X et K_C satisfaisant les Hypothèses 1.2.27 avec $L_X, L_C < \mathcal{C}_{K_X,K_C}$, il existe deux constantes B_1 et B_2 positives et explicites telles que pour toute mesure de probabilité μ_0 sur $\mathbb{R}^2$ satisfaisant $\mathbb{E}_{\mu_0}\left(e^{\tilde{a}(|X|+|C|)}\right) \leq \tilde{\mathcal{C}^0}$, on a*

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq B_1 \frac{k}{\sqrt{N}}, \qquad \mathcal{W}_2^2\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq B_2 \frac{k}{\sqrt{N}},$$

pour tout $k \in \mathbb{N}$, où $\mu_t^{k,N}$ est la distribution marginale, à l'instant t, des k premiers neurones $(Z_t^{1,N}, \dots, Z_t^{k,N})$ d'un système de N neurones suivant le système (1.2.23), avec pour distribution initiale $(\mu_0)^{\otimes N}$, et où $\bar{\mu}_t$ est une solution de (1.2.24) avec pour distribution initiale μ_0.

1.2.2.3 Stratégies de preuve

Afin de prouver qu'il y a des solutions bien définies pour (1.2.23) (première partie de la Proposition 1.2.28), nous approchons la fonction cubique $x \mapsto x^3$ par une fonction localement Lipschitz et bornée g_K :

$$g_K(x) = \begin{cases} -K^3 & \text{if } x < -K \\ x^3 & \text{if } x \in [-K, K] \\ K^3 & \text{if } x > K. \end{cases}$$

Les théorèmes usuelles peuvent alors s'appliquer, et à l'aide d'une fonction de Lyapunov H, quadratique, nous prouvons que les solutions de nos systèmes n'explosent pas.
La seconde partie de cette Proposition, c'est-à-dire l'existence et l'unicité d'une solution de (1.2.24), repose sur un théorème de [70] dont nous vérifions les hypothèses.

Pour démontrer la convergence de $\mu_t^{k,N}$ vers $\bar{\mu}_t^{\otimes k}$ quand N tend vers l'infini, nous suivons la méthode de couplage décrite dans [40]. Le résultat de cet article ne peut s'appliquer directement en raison de la dynamique cubique de notre processus. Cette méthode a été suggérée par Eberle [31], à la suite de travaux de Lindvall et Rogers [57], et a été approfondie par le travail de [29].

Une distance naturelle entre deux mesures de probabilité est la distance de Wasserstein, liée à la théorie de transport optimale. Nous rappelons qu'elle se définit de la manière suivante : pour μ et ν deux mesures de probabilité sur $\mathbb{R}^d$, on note

$$\mathcal{W}_p(\mu, \nu) = \inf_{X \sim \mu,\ Y \sim \nu} \mathbb{E}\left(||X - Y||_p^p\right)^{1/p}, \tag{1.2.25}$$

où $||\cdot||_p$ décrit la distance usuelle L^p sur $\mathbb{R}^d$. Cette distance étant définie comme le minimum de l'espérance de la distance entre X et Y sur tous les couplages de (X, Y) où X suit la loi μ et Y suit la loi ν, il suffit de construire un couplage telle que cette quantité tend vers 0. Ainsi, nous construisons simultanément deux solutions de

(1.2.23) et (1.2.24) qui se rapprocheront lorsque le nombre de neurones augmentera.

Soit $\left(\bar{X}_t^i, \bar{C}_t^i\right)$, pour i entre 1 et N, N copies indépendantes de solutions de (1.2.24) dirigées par des mouvements Browniens $(\bar{B}_t^{i,X})_{t\geqslant 0}$ et $(\bar{B}_t^{i,C})_{t\geqslant 0}$ indépendants. Un couplage de deux jeux de solutions $\left(\bar{X}_t^i, \bar{C}_t^i\right)_i$ et $\left(X_t^{i,N}, C_t^{i,N}\right)_i$ découle alors directement d'un couplage des mouvements Browniens B et $\bar{B}$.

Le premier choix, le plus naturel, a été popularisé par Sznitman [77]. Il s'agit du couplage synchrone qui consiste à choisir $B = \bar{B}$. Ainsi, quand on considère l'évolution temporelle de $\bar{Z}_t^i - Z_t^{i,N} = \left(\bar{X}_t^i - X_t^{i,N}, \bar{C}_t^i - C_t^{i,N}\right)$, le bruit s'annule. La contraction de la distance entre les processus repose alors uniquement sur la dérive (ou le drift) déterministe. Même s'il est, en général, alors nécessaire d'avoir une dérive fortement convexe, dans notre cadre, cette approche suffit pour démontrer une propagation du chaos non uniforme en temps. On peut également voir qu'il y a une contraction déterministe suffisante lorsque $\bar{X}_t^i - X_t^{i,N} = 0$. On va alors utiliser un couplage synchrone dans le voisinage de ce sous-espace.

En dehors de ce sous-espace, nous utilisons le bruit pour rapprocher les processus. On considère $B = -\bar{B}$, qui maximise la variance du bruit, dans la direction orthogonale à ce sous-espace. Il s'agit alors d'un couplage par réflexion (également nommé antithétique). Néanmoins, comme le bruit est symétrique, si nous considérons une distance classique de type $r_t^i = |\bar{X}_t^i - X_t^{i,N}| + \delta|\bar{C}_t^i - C_t^{i,N}|$ avec $\delta > 0$, il n'y a pas de raison pour que ce couplage favorise une décroissance de r_t^i. C'est pourquoi nous regardons $f(r_t^i)$, avec f une fonction concave spécifique. Une diminution aléatoire a alors plus d'effet qu'une augmentation aléatoire de même valeur.

Finalement, nous pouvons définir une fonction de Lyapunov $\tilde{H}$ pour prendre en compte la tendance de chaque processus à revenir dans un ensemble compact de $\mathbb{R}^2$. Nous étudions ensuite une distance adaptée entre les processus, qui dépend de la fonction de Lyapunov afin de prendre en compte la structure du système de particules. Cette distance est de la forme $\rho_t := \frac{1}{N}\sum_{i=1}^N f(r_t^i)(1+\epsilon\tilde{H}(\bar{Z}_t^i)+\epsilon\tilde{H}(Z_t^{i,N}))$, avec $\epsilon > 0$. Cette quantité permet de contrôler les distances L^1 et L^2 usuelles. Lorsque r_t^i est petit, $f(r_t^i)$ va décroître, soit grâce à la partie déterministe, soit grâce au couplage par réflexion. Lorsque r_t^i est grand, c'est la partie avec la fonction de Lyapunov $\tilde{H}$ qui va décroître. Cela nous permet de montrer que $\mathbb{E}(\rho_t)$ décroît exponentiellement vite. Bien sûr, cela mène à un certain nombre de contraintes sur les différents paramètres et nous vérifions que ces contraintes peuvent bien être satisfaites. La distance ρ_t sera légèrement modifiée pour prendre en compte la non-linéarité du processus.

La méthode décrite ci-dessus a bien sûr besoin de bruit dans la direction orthogonale au sous-espace naturellement contractant. Ainsi, il est nécessaire d'avoir $\sigma_X > 0$. Dans le cas où $\sigma_X = 0$ et $\sigma_C > 0$, un autre sous-espace est considéré et les calculs sont en conséquence modifiés. Ce cas est traité en annexe.

1.2.2.4 Perspectives

Pour conclure, le résultat principal de ce chapitre est une propagation du chaos uniforme en temps dans le modèle de FitzHugh-Nagumo, et est démontrée à partir d'un couplage mixte entre synchrone et antithétique. Nous avons conservé un modèle assez général, en envisageant un bruit et une interaction sur chacune des coordonnées. Cette méthode de couplage nous a également permis de définir des constantes de vitesse explicites.

Comme cela a été présenté rapidement dans l'introduction, de nombreuses questions sont soulevées pour les modèles neuronaux, aussi ce travail a de nombreuses directions d'extensions. Tout d'abord, certains auteurs [74] se sont intéressés à la notion de bruit dit environnemental dans d'autres cadres. Il s'agit d'un bruit identique pour chacune des particules : l'idée est d'avoir $B^{i,X} = B^X$ et $B^{i,C} = B^C$. Nous pourrions nous intéresser au comportement à long-terme de ce type de processus, ou de processus présentant à la fois un bruit environnemental et un bruit individuel.

Par ailleurs, [59] et [13] s'est concentré sur des propriétés de synchronisation neuronales : il a été observé, biologiquement, que certains neurones se synchronisent et se déchargent de manière groupée, les décharges ayant alors lieu sur des plages de temps assez réduites. Certains modèles permettent de modéliser cette synchronisation, en particulier le modèle de FitzHugh-Nagumo, dans un cadre dans lequel la dynamique du drift est "ralenti" devant la dynamique générée par l'interaction et le bruit.

Nous pourrions également étendre notre étude en considérant des paramètres non plus globaux mais individuels. On peut envisager des paramètres α^i, β^i et γ^i pour chacun des neurones i, qui suivent une certaine répartition autour de paramètres moyens α, β et γ. Les mêmes questions que ci-dessus pourraient alors être posées : la synchronisation étudiée dans [59] est vraie pour des paramètres appartenant à un certain ensemble. Qu'en serait-il si le jeu de paramètres d'une partie des neurones est dans cet ensemble, et celui du reste des neurones n'y est pas ? Aurait-on un entraînement généré par les neurones synchronisés qui se propagerait aux autres ? Si c'est le cas, à partir de quelle proportion de neurones dans le "bon" ensemble de paramètres ?

L'article [4], complété par [12], considère des interactions plus élaborées pour les processus de FitzHugh-Nagumo. Ici, comme beaucoup dans la littérature, nous nous sommes focalisés sur des synapses dites électriques, qui peuvent être représentées comme une fonction (K_X ou K_C) de la différence entre deux états $Z^i - Z^j$. Les synapses chimiques nécessitent une autre équation dans le système d'EDS, ce qui augmente la complexité du modèle.
Nous voudrions également généraliser notre étude en ne nous limitant pas à des interactions Lipschitz, et en considérant par exemple des fonctions discontinues qui

peuvent représenter des effets seuils, ou des interactions ne prenant pas en compte l'état du neurone i étudié (qui ne dépendraient que de l'état des Z^j).

Enfin, si ici nous nous sommes concentrés sur le modèle FitzHugh-Nagumo à 2 EDS par neurone, nous pourrions explorer le modèle Hodgkin-Huxley à 4 EDS par neurone (voire plus si des synapses chimiques sont prises en compte) et vérifier si les outils mis en place ici peuvent s'envisager. Par exemple, [4] et [12] traitent à la fois le modèle FitzHugh-Nagumo et le modèle Hodgkin-Huxley.

Chapter 2

Large deviation principles for cumulative processes

This chapter consists in the article [19] submitted, and available on arXiv. Notice that the companion paper [20] cited several times is the Chapter 3.
Some proofs, very similar to [52] were not included in this article. We have added them in the Appendix 2.A. Some are exactly the same, others are developed, and others are slightly different since the framework is different.

Abstract

The aim of this paper is to prove a Large Deviation Principle (LDP) for cumulative processes also known as coumpound renewal processes. These processes cumulate independent random variables occuring in time interval given by a renewal process. Our result extends the one obtained in [52] in the sense that we impose no specific dependency between the cumulated random variables and the renewal process. The proof is inspired from [52] but deals with additional difficulties due to the general framework that is considered here. In the companion paper [20] we apply this principle to Hawkes processes with inhibition. Under some assumptions Hawkes processes are indeed cumulative processes, but they do not enter the framework of [52].

Contents

2.1 Introduction.

2.1.1 Cumulative processes

Cumulative processes have been introduced by Smith [76] and are applied in many purposes, such as finance where they are called *compound-renewal* processes or *renewal-reward* processes. Indeed these continuous time processes cumulate independent random variables occurring in time interval given by a renewal process. To be more specific a real valued process $(Z_t)_{t\geq 0}$ is called *a cumulative process* if the following properties are satisfied:

1. $Z_0 = 0$,
2. there exists a renewal process $(S_n)_n$ such that for any n, $(Z_{S_n+t} - Z_{S_n})_{t\geq 0}$ is independent of $S_0, ...S_n$ and $(Z_s)_{s<S_n}$,
3. the distribution of $(Z_{S_n+t} - Z_{S_n})_{t\geq 0}$ is independent of n.

To study such processes, we write for all $t \geq 0$

$$Z_t = W_0(t) + W_1 + ... + W_{M_t} + r_t$$

where $W_0(t) = Z_{t\wedge S_0}$, $W_n = Z_{S_n} - Z_{S_{n-1}}$, $r_t = Z_t - Z_{M_t}$, where M_t is defined by

$$M_t = \sup\{n \geq 0, S_n \leq t\}.$$

The $(W_k)_{k\geq 1}$'s are thus i.i.d.

We denote by $(\tau_i)_i$ the times associated to the renewal process $\tau_n = S_n - S_{n-1}$ and $\tau_0 = 0$.

It is worth noticing that τ_i and W_i can be dependent.

In the sequel we suppress the subscript i when dealing with the distribution (and all associated quantities like expectation, variance ...) of (τ_i, W_i) and simply use (τ, W).

A simple example of cumulative process is $Z_t = \int_0^t f(X_s)ds$ where $(X_t)_t$ is a regenerative process with i.i.d. cycles [37]. Markov additive processes are other classical examples of cumulative process.

In [22] the authors exhibited a renewal structure for some Hawkes processes. This description is extensively used in our companion paper [20] in order to describe such processes as cumulative processes, and to study their asymptotic behaviour.

For cumulative processes, the law of large numbers (assuming that $\mathbb{E}[|W|]$ and $\mathbb{E}[\tau]$ are not infinite)

$$\frac{Z_t}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]} \text{ if and only if } \mathbb{E}\left(\max_{S_0 \le t < S_1} |r_t|\right) < \infty\,,$$

and the central limit theorem (assuming $\mathrm{Var}(W) < \infty$ and $\mathrm{Var}(\tau) < \infty$)

$$\frac{\left(Z_t - t\frac{\mathbb{E}[W]}{\mathbb{E}[\tau]}\right)}{\sqrt{t}} \underset{t\to\infty}{\longrightarrow} \mathcal{N}\left(0, \sigma^2\right) \text{ where } \sigma^2 = \frac{1}{\mathbb{E}(\tau)}\mathrm{Var}\left(W - \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]}\tau\right)$$

can be found in Asmussen [1], theorem 3.1 and theorem 3.2.
Brown and Ross [16] have proved equivalent of the Blackwell theorem and of the key renewal theorem for a subclass of cumulative processes, since cumulative processes are a generalization of renewal processes. Glynn and Whitt have focused in [37] on cumulative processes associated to a regenerative process and have proved law of large numbers (strong and weak), law of the iterated logarithm, central limit theorem and functional generalizations of these properties.

The aim of this work is to establish a large deviation principle (LDP) for cumulative processes. Some works have already been done. Duffy and Metcalfe [28] have considered the estimation of a rate function for a cumulative process (if it admits a LDP). In a series of papers, Borovkov and Mogulskii ([9, 10, 11]) have studied the (LDP) (they use the term compound-renewal theorem), under some assumptions of comparison between the values (θ_1, θ_2) for which $\mathbb{E}\left(e^{\theta_1\tau+\theta_2 W}\right)$ is finite and the value θ for which $\mathbb{E}\left(e^{\theta\tau}\right)$ is finite (a comparison between the joint distribution of (τ, W) and the law of τ). Actually, some points in their approach are not clear for us.

Lefevere, Mariani and Zambotti [52] worked on specific cumulative process where $W_i = F(\tau_i)$ for some deterministic function F which is assumed to be non-negative, bounded and continuous. We choose to extend their approach. Our proof of the LDP has the same skeleton, but in a general framework for the pair (τ, W).

In this paper, we show a LDP for Z_t/t in the case $r_t = 0$. This assumption can be relaxed if r_t/t tends to 0 quickly enough, as it will be the case for the application to Hawkes process (see [20]). For example, if for all $\delta > 0$

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{|r_t|}{t} > \delta\right) = -\infty,$$

then Z_t/t and $(Z_t - r_t)/t$ are exponentially equivalent and satisfy the same LDP (if $(Z_t - r_t)/t$ have one) and have the same rate function.

2.1.2 Hawkes processes

A Hawkes process is a point process on the real line $\mathbb{R}$ characterized by its intensity process $t \mapsto \Lambda(t)$. We consider an appropriate filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual assumptions.

Definition 2.1.1. Let $\lambda > 0$ and $h : (0, +\infty) \to \mathbb{R}$ a signed measurable function. Let N^0 a locally finite point process on $(-\infty, 0]$ with law $\mathbf{m}$.
The point process N^h on $\mathbb{R}$ is a Hawkes process on $(0, +\infty)$, with initial condition N^0 and reproduction measure $\mu(dt) = h(t)dt$ if:

— $N^h\,|_{(-\infty,0]} = N^0$,

— the conditional intensity measure of $N^h\,|_{(0,+\infty)}$ with respect to $(\mathcal{F}_t)_{t\geq 0}$ is absolutely continuous w.r.t the Lebesgue measure and has density:

$$\Lambda^h : t \in (0, +\infty) \mapsto f\left(\lambda + \int_{(-\infty,t)} h(t-u)N^h(du)\right). \tag{2.1.1}$$

for some non-negative function f.

Hawkes processes have been introduced by Hawkes [43]. Most of the literature concerned with the large time behaviour of $N_t^h = N^h([0,t])$ is dedicated to the case $h \geq 0$ (self excitation). This behaviour is studied in details in [20] when h is a signed (the negative part modelling self inhibition) compactly supported function, and the function f (called the jump rate function) is given by

$$f(u) = \max(0, u)\,.$$

In this situation one gets a description of N_t^h as a cumulative process (see [20] subsection 2.3) with few information on the joint law of (τ, W). This was the initial motivation for the present work. We refer to [20] for a more complete overview and explicit results in this situation. In the remaining part of the paper we will come back to more general cumulative processes.

2.2 Notations and main result

2.2.1 First notations.

We consider $(\tau_1, W_1), (\tau_2, W_2), \ldots$ an i.i.d. sequence of pairs of random variables built on some probability space $(\Omega, \mathbb{P})$. The law of (τ_i, W_i) is an arbitrary probability

ψ on $(0,+\infty)\times\mathbb{R}$. We denote this by: $(\tau_i, W_i) \sim \psi$. In the sequel we generically use the notation (τ, W) for a pair with the same distribution as (τ_i, W_i). Notice that we assume in particular that $\mathbb{E}(\tau) > 0$.

We denote by $\mathcal{M}^1(\mathcal{X})$ the space of probability measure on some measurable space $\mathcal{X}$.

We consider the renewal process associated with $(\tau_i)_{i\geq 1}$:

$$S_0 = 0, \ \ S_n = \sum_{k=1}^{n} \tau_k,$$
$$M_t = \sup\{n \geq 0, S_n \leq t\}.$$

We will study the quantity:

$$Z_t = \sum_{i=1}^{M_t} W_i. \tag{2.2.1}$$

The main goal of this paper is to prove a Large Deviation Principle for the process Z_t/t. Let us recall some basic definitions in large deviation theory (we refer to [52]).

A family of probability measures $(\eta_t)_{t\in\mathbb{R}^+}$ on a topological space $(\mathcal{X}, T_{\mathcal{X}})$ satisfies the Large Deviations Principle (LDP) with rate function $J(.)$ and speed $\gamma(t) = t$ if J is lower semi-continuous from $\mathcal{X}$ to $[0,+\infty]$, and the following holds

$$-\inf_{x\in\mathcal{O}} J(x) \leq \liminf_{t\to+\infty} \frac{1}{t} \ln \eta_t(\mathcal{O}) \quad \text{for all open subset } \mathcal{O}, \tag{2.2.2}$$

and

$$-\inf_{x\in\mathcal{C}} J(x) \geq \limsup_{t\to+\infty} \frac{1}{t} \ln \eta_t(\mathcal{C}) \quad \text{for all closed subset } \mathcal{C}. \tag{2.2.3}$$

We shall sometimes say that $(\eta_t)_{t\in\mathbb{R}^+}$ satisfies the *full* LDP when (2.2.2) and (2.2.3) are satisfied, while we will use *weak* LDP when $\mathcal{C}$ closed is replaced by $\mathcal{C}$ compact in (2.2.3). When η_t is the distribution of some random variable Y_t (for instance Z_t/t) we shall say that the family $(Y_t)_t$ satisfies a LDP.

Since J is lower semi-continuous the level sets $\{X, J(x) \leq a\}$ are closed. If in addition they are compact, then J is said to be a good rate function.

In this paper we only consider the speed function $\gamma(t) = t$ so that we will no more refer to it.

A particularly important notion for our purpose is the notion of *exponentially good approximation.*

Definition 2.2.1. Assume that $(\mathcal{X}, d)$ is a metric space. A family of random variables $Y_{n,t}$ $(n \in \mathbb{N})$ is an exponentially good approximation of Y_t (all these variables being defined on the same probability space $(\Omega, \mathbb{P})$), if for all $\delta > 0$ it holds

$$\lim_{n\to\infty} \limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}(d(Y_{n,t}, Y_t) > \delta) = -\infty.$$

The key result is then

Theorem 2.2.2. In the framework of definition 2.2.1, assume that $Y_{n,t}$ is an exponentially good approximation of Y_t. Then the following statements hold true.

1. If $Y_{n,t}$ satisfies a full LDP with rate function J^n then Y_t satisfies a weak LDP with rate function
$$J(x) = \sup_{\delta>0} \liminf_{n\to\infty} \inf_{d(y,x)<\delta} J^n(y)\,.$$
2. If $\mathcal{X} = \mathbb{R}^k$ equipped with any norm, then the same conclusion is true when $Y_{n,t}$ satisfies only a weak LDP.
3. If J (defined above) is a good rate function such that for any closed set F,
$$\inf_{y\in F} J(y) \le \limsup_{n\to\infty} \inf_{y\in F} J^n(y)$$
then Y_t satisfies a full LDP with rate function J.

The first and last points in the previous Theorem are contained in [52] Theorem 4.2.16. The second one is a consequence of the fact that closed balls are compact sets in $\mathbb{R}^k$. Usually, the Theorem is sufficient to prove a full LDP. Nevertheless, it some cases, the study of the rate function J is difficult. The lemma below gives an alternative, using exponential tightness which is easy to obtain with our assumptions.

Lemma 2.2.3. If Y_t satisfies a weak LDP with a rate function I and is exponentially tight, i.e. for all $\alpha > 0$, there exists a compact set K_α such that
$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(Y_t \notin K_\alpha^c\right) < -\alpha,$$
then Y_t satisfies a full LDP and I is a good rate function.

This Lemma is a consequence of the Lemma 1.2.18 in [52].
Theorem 2.2.2 as well as Lemma 2.2.3 will be used at the end of the article, in the section 2.9 which proves the main Theorem 2.2.5.

2.2.2 Main results.

Assumption 2.2.4. We will make the following set of assumptions:
i) $\exists \theta_0 \in (0,+\infty]$ such that $\mathbb{E}[\mathrm{e}^{\theta\tau}] < \infty$ for $\theta < \theta_0$,
ii) $\exists \beta_0 \in (0,+\infty]$ such that $\mathbb{E}[\mathrm{e}^{\beta|W|}] < \infty$, for $\beta < \beta_0$.

We will introduce the classical Cramer transforms, for $(a,b) \in \mathbb{R}^2$,

$$\Lambda^*(a,b) = \sup_{x,y} \{ax + by - \ln \mathbb{E}(e^{x\tau+yW})\} \tag{2.2.4}$$

$$\Lambda_n^*(a,b) = \sup_{x,y} \left\{ax + by - \ln \mathbb{E}\left(e^{x\tau+yW^n}\right)\right\}\,, \tag{2.2.5}$$

where W^n is a well-chosen reduction of W. We finally introduce the rate functions

$$J^n(m) = \inf_{\beta>0} \beta\,\Lambda_n^*\left(\frac{1}{\beta},\frac{m}{\beta}\right) \quad , \quad J(m) = \inf_{\beta>0} \beta\,\Lambda^*\left(\frac{1}{\beta},\frac{m}{\beta}\right), \tag{2.2.6}$$

and

$$\tilde{J} = \sup_{\delta>0} \liminf_{n\to\infty} \inf_{|m-z|<\delta} J^n(z)\,. \tag{2.2.7}$$

We then may state

Theorem 2.2.5. Assume that Assumption 2.2.4 is fulfilled. Let J given by (2.2.6). We denote $m = \mathbb{E}(W)/\mathbb{E}(\tau)$.

— If $\beta_0 = +\infty$ (in particular if W is bounded) then Z_t/t satisfies a full LDP with good rate function $\tilde{J}$. We also have the following inequalities

$$\limsup_{t\to+\infty} \frac{1}{t}\ln\mathbb{P}\left(\frac{Z_t}{t} \geq m+a\right) \leq -\inf_{z\geq m+a} J(z), \tag{2.2.8}$$

$$\limsup_{t\to+\infty} \frac{1}{t}\ln\mathbb{P}\left(\frac{Z_t}{t} \leq m-a\right) \leq -\inf_{z\leq m-a} J(z). \tag{2.2.9}$$

— If $\beta_0 < +\infty$, we have for all $a > 0$

$$\limsup_{t\to+\infty} \frac{1}{t}\ln\mathbb{P}\left(\frac{Z_t}{t} \geq m+a\right) \leq -\min\left[\inf_{z\geq m+(a/2)} J(z)\;,\;\beta_0 a/4\right], \tag{2.2.10}$$

and

$$\limsup_{t\to+\infty} \frac{1}{t}\ln\mathbb{P}\left(\frac{Z_t}{t} \leq m-a\right) \leq -\min\left[\inf_{z\geq m-(a/2)} J(z)\;,\;\beta_0 a/4\right]. \tag{2.2.11}$$

We may replace $a/2$ and $\beta_0/2$ by κa and $(1-\kappa)\beta_0$ for any $\kappa \in (0,1)$.

Remark 2.2.6. The previous inequalities (2.2.8) and (2.2.9) actually hold true with function $\tilde{J}$ as well as function J since $J \leq \tilde{J}$ (see Lemma 2.9.5). However, since J is clearly more easy to calculate than $\tilde{J}$, we prefer to write the inequalities with J.

2.2.3 The scheme of proof.

The proof will be divided in several steps. In the next section we will reduce the problem first to bounded W's replacing the original W by W^n, and then to finite valued W's. This will be done by using exponentially good approximations and Theorem 2.2.2.

Following [52] we introduce in section 2.4 (for finite valued W's) an associated empirical measure and state the LDP for this measure (theorem 2.4.3) and the LDP for the cumulative process (theorem 2.4.4), the latter being obtained via the contraction principle. The following sections are devoted to the proofs of these theorems.

Section 2.5 is devoted to the study of the rate function in theorem 2.4.3. This is the most technical part of this work. The idea is that one can adapt the arguments in [52] by conditioning w.r.t. each value of W.

At this point the fact that W is finite valued will provide us both with simple conditioning and with the necessary compactness arguments, since $\mathcal{M}_1(\{1,...,n\})$ is compact.

If many arguments are close to those in [52], some have to be written in detail. We have decided to refer to the corresponding statements in [52] only when they can be reproduced line by line in our case. Some topological points also have to be clarified. In section 2.6 we show useful auxiliary lemmata. The next section 2.7 is devoted to the proof of 2.4.3. In section 2.8 we deduce theorem 2.4.4. The final section 2.9 is devoted to the study of the rate function J and to the proof of theorem 2.2.5.

2.3 Reduction of the problem to finite valued W's.

2.3.1 First reduction to bounded W's.

To Z_t defined by (2.2.1) we associate

$$Z_t^n = \sum_{i=1}^{M_t} (W_i \wedge n \vee (-n)) \,,$$

so that

$$|Z_t - Z_t^n| = \sum_{i=1}^{M_t} (W_i - n)_+ + \sum_{i=1}^{M_t} (W_i + n)_- \tag{2.3.1}$$

where $u_+ = \max(u, 0)$ and $u_- = \max(-u, 0)$. We then have

Lemma 2.3.1. If Assumption 2.2.4 is fulfilled, for all $\delta > 0$,

$$\lim_{n\to\infty} \limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{Z_t^n}{t}\right| > \delta\right) \le -\frac{\beta_0\,\delta}{2}\,.$$

In particular if $\beta_0 = +\infty$, Z_t^n/t is an exponentially good approximation of Z_t/t.

Proof. It is enough to look at

$$\mathbb{P}\left(\sum_{i=1}^{M_t} (W_i - n)_+ > \delta t\right)\,,$$

since the other term in (2.3.1) can be treated similarly. To conclude it is enough to use the elementary

$$\ln(a + b) \le \max(\ln(2a), \ln(2b))\,. \tag{2.3.2}$$

Using that the W_i's are i.i.d. we may write for $\delta > 0$ and $c > 0$, (as usual an empty sum is equal to 0 by convention)

$$
\begin{aligned}
&\mathbb{P}\left(\sum_{i=1}^{M_t}(W_i - n)_+ > \delta t\right)\\
&\leq \mathbb{P}\left(\sum_{i=1}^{ct}(W_i - n)_+ > \frac{\delta t}{2}\right) + \mathbb{P}\left(\sum_{i=ct+1}^{M_t}(W_i - n)_+ > \frac{\delta t}{2}\right)\\
&\leq \mathbb{P}\left(\sum_{i=1}^{ct}(W_i - n)_+ > \frac{\delta t}{2}\right) + \mathbb{P}\left(\left\{\sum_{i=ct+1}^{M_t}(W_i - n)_+ > \frac{\delta t}{2}\right\} \cap \{1 + ct \leq M_t < 2ct\}\right)\\
&\qquad + \mathbb{P}\left(\left\{\sum_{i=ct+1}^{M_t}(W_i - n)_+ > \frac{\delta t}{2}\right\} \cap \{M_t \geq 2ct\}\right)\\
&\leq 2\mathbb{P}\left(\sum_{j=1}^{ct}(W_j - n)_+ > \frac{\delta t}{2}\right) + \mathbb{P}\left(M_t \geq 2ct\right)
\end{aligned}
$$

Study of $\mathbb{P}\left(M_t \geq 2ct\right)$. Start with the second term in the sum above. According to theorem 2.3 in [80], we know that M_t/t satisfies a LDP with rate function J_τ given by

$$
J_\tau(x) = \begin{cases} \sup_\lambda\{\lambda - x \ln \mathbb{E}(\mathrm{e}^{\lambda\tau})\} & \text{if } x \geq 0 \\ \infty & \text{if } x < 0 \end{cases}
$$

Notice that $J_\tau(x) = x\,\Lambda^*(1/x, 0)$ for $x > 0$. In addition (see Lemma 2.6 in [80]) the supremum is achieved for $\lambda \leq 0$ if $x \in (1/\mathbb{E}(\tau)\,, +\infty)$ and J_τ is non-decreasing on this interval.

It follows that for $2c > 1/\mathbb{E}(\tau)$,

$$
\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(M_t \geq 2ct\right) = -J_\tau(2c)\,. \tag{2.3.3}
$$

In order to get $\lim_n \limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(M_t \geq 2c_n t\right) = -\infty$ for some sequence c_n (to be chosen later) it remains to show that

$$
J_\tau(x) \underset{x\to\infty}{\longrightarrow} +\infty.
$$

Since τ is a non-negative random variable one can find λ_τ such that

$$
\mathbb{E}(\mathrm{e}^{\lambda_\tau\tau}) = \mathrm{e}^{-1}\,.
$$

Let $x \in \mathbb{R}^+$, we have:

$$
\begin{aligned}
J_\tau(x) &= \sup_\lambda\{\lambda - x \ln \mathbb{E}(\mathrm{e}^{\lambda\tau})\}\\
&\geq \lambda_\tau - x \ln \mathbb{E}(\mathrm{e}^{\lambda_\tau\tau})\\
&\geq x + \lambda_\tau
\end{aligned}
$$

yielding the desired result.

Study of $\mathbb{P}\left(\sum_{j=1}^{ct}(W_j-n)_+ > \frac{\delta t}{2}\right)$**.** Denote as usual by $\lfloor ct\rfloor$ the integer part of ct. We have

$$\mathbb{P}\left(\sum_{j=1}^{ct}(W_j-n)_+ > \frac{\delta t}{2}\right) \quad = \quad \mathbb{P}\left(\sum_{j=1}^{\lfloor ct\rfloor}(W_j-n)_+ > \frac{\delta t}{2}\right)$$

so that we may use this time the usual Cramer's theorem. Defining

$$\Psi_n(\lambda) = \ln \mathbb{E}\left[\mathrm{e}^{\lambda(W-n)_+}\right]$$
$$\Psi_n^*(x) = \sup_\lambda\{\lambda x - \Psi_n(\lambda)\},$$

we have

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\ln\mathbb{P}\left(\sum_{j=1}^{\lfloor ct\rfloor}(W_j-n)_+ > \delta t/2\right) \quad &= \quad \limsup_{t\to\infty}\frac{c}{\lfloor ct\rfloor}\ln\mathbb{P}\left(\sum_{j=1}^{\lfloor ct\rfloor}(W_j-n)_+ > \delta t/2\right)\\ &\le \quad \limsup_{t\to\infty}\frac{c}{\lfloor ct\rfloor}\ln\mathbb{P}\left(\sum_{j=1}^{\lfloor ct\rfloor}(W_j-n)_+ > \delta\lfloor ct\rfloor/2c\right)\\ &\le \quad -c\inf_{x\in[\delta/2c,+\infty)}\Psi_n^*(x).\end{aligned}$$

As the function $x\mapsto\Psi_n^*(x)$ is non-decreasing on $[\mathbb{E}((W-n)_+),+\infty)$, we have

$$\limsup_{n\to\infty}\frac{1}{t}\ln\mathbb{P}\left(\sum_{j=1}^{\lfloor ct\rfloor}(W_j-n)_+ > \delta t/2\right) = -\,c\,\Psi_n^*(\delta/2c)\,,$$

provided $\delta/2c \ge \mathbb{E}((W-n)_+)$. Notice that for $\lambda<\beta_0$,

$$c\,\Psi_n^*(\delta/2c) \;\ge\; \frac{\lambda\delta}{2} \;-\; c\,\ln\left(1+\mathbb{E}\left[(e^{\lambda(W-n)}-1)\,\mathbb{1}_{W>n}\right]\right)$$

so that choosing c_n growing to infinity and such that

$$c_n\,\mathbb{E}((W-n)_+)\to 0 \;\text{ and }\; c_n\,\ln\left(1+\mathbb{E}\left[(e^{\lambda(W-n)}-1)\mathbb{1}_{W>n}\right]\right)\to 0 \text{ as } n\to\infty$$

which is always possible since both $\mathbb{E}((W-n)_+)$ and $\ln\left(1+\mathbb{E}\left[(e^{\lambda(W-n)}-1)\mathbb{1}_{W>n}\right]\right)$ are going to 0, we get

$$\lim_n\;\limsup_{t\to\infty}\frac{1}{t}\ln\mathbb{P}\left(\sum_{j=1}^{\lfloor ct\rfloor}(W_j-n)_+ > \delta t/2\right) \ge \frac{\lambda\delta}{2}\,.$$

We may optimize in λ and plugging the same sequence c_n in (2.3.3) ends the proof. □

2.3.2 Second reduction to finite valued W's.

Starting with a bounded W such that $-K < W < K$ almost surely we define a discretized version of W by

$$W^n = \sum_{j=-n}^{n-1} \frac{jK}{n} \mathbb{1}_{W \in [jK/n,(j+1)K/n[} .$$

It clearly holds

$$|W - W^n| \leq \frac{K}{n} .$$

We thus introduce

$$\bar{Z}_t^n = \sum_{i=1}^{M_t} W_i^n$$

so that

$$|\bar{Z}_t^n - Z_t| \leq \frac{K\, M_t}{n} .$$

According to the study in the previous section

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\left|\frac{\bar{Z}_t^n}{t} - \frac{Z_t}{t}\right| > \delta\right) \leq \limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(M_t \geq \delta t n/K\right) = -J_\tau(\delta n/K)$$

so that, since J_τ grows to infinity

$$\lim_{n\to\infty} \limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\left|\frac{\bar{Z}_t^n}{t} - \frac{Z_t}{t}\right| > \delta\right) = -\infty .$$

We have thus shown

Lemma 2.3.2. If Assumption 2.2.4 is fulfilled, and W is almost surely bounded, $\bar{Z}_t^n/t$ defined above is an exponentially good approximation of Z_t/t.

Remark 2.3.3. If W isn't bounded nor discrete, a double reduction can be done in one step. We obtain the same results as doing successively both the reductions, but it allows formulating the rate function more easily, when there is one.

Let's consider

$$\tilde{W}^n = -n\mathbb{1}_{W<-n} + n\mathbb{1}_{W\geq n} + \sum_{j=-n^2}^{n^2-1} \frac{j}{n} \mathbb{1}_{W\in[\frac{j}{n},\frac{j+1}{n})} .$$

By denoting $\tilde{Z}_t^n = \sum_{i=1}^{M_t} \tilde{W}_i^n$, we have, for $\delta > 0$

$$\begin{aligned}\mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{\tilde{Z}_t^n}{t}\right| > \delta\right) \leq & \mathbb{P}\left(\sum_{i=1}^{M_t} (W_i - n)_+ > \delta t\right) + \mathbb{P}\left(\sum_{i=1}^{M_t} (W_i + n)_+ > \delta t\right) \\ & + \mathbb{P}\left(\sum_{i=1}^{M_t} \sum_{j=-n^2}^{n^2-1} \left(W_i - \frac{j}{n}\right) \mathbb{1}_{W_i\in[\frac{j}{n},\frac{j+1}{n})} > \delta t\right) .\end{aligned}$$

Since

$$\left|\sum_{i=1}^{M_t}\sum_{j=-n^2}^{n^2-1}\left(W_i-\frac{j}{n}\right)\mathbb{1}_{W_i\in[\frac{j}{n},\frac{j+1}{n})}\right|\leq\frac{M_t}{n},$$

we obtain

$$\mathbb{P}\left(\sum_{i=1}^{M_t}\sum_{j=-n^2}^{n^2-1}\left(W_i-\frac{j}{n}\right)\mathbb{1}_{W_i\in[\frac{j}{n},\frac{j+1}{n})}>\delta t\right)\leq\mathbb{P}(M_t>\delta tn).$$

By the proofs of Lemmas 2.3.1 and 2.3.2, we finally obtain

Lemma 2.3.4. If Assumption 2.2.4 is fulfilled, for all $\delta>0$,

$$\lim_{n\to\infty}\limsup_{t\to\infty}\frac{1}{t}\ln\mathbb{P}\left(\left|\frac{Z_t}{t}-\frac{\tilde{Z}_t^n}{t}\right|>\delta\right)\leq-\frac{\beta_0\,\delta}{2}.$$

In particular if $\beta_0=+\infty$, $\tilde{Z}_t^n/t$ is an exponentially good approximation of Z_t/t.

Remark 2.3.5. The proofs in the previous section suggest a direct naive approach in order to get deviation bounds. Indeed we may write

$$\left|\frac{Z_t}{t}-\frac{\mathbb{E}(W)}{\mathbb{E}(\tau)}\right|\leq\frac{1}{t}\left|\sum_{i=1}^{\lfloor M_t\rfloor}(W_i-\mathbb{E}(W))\right|+\mathbb{E}(W)\left|\frac{\lfloor M_t\rfloor}{t}-\frac{1}{\mathbb{E}(\tau)}\right|$$

so that following the lines of the proof of lemma 2.3.1, we may write

$$\begin{aligned}
&\mathbb{P}\left(\left|\frac{Z_t}{t}-\frac{\mathbb{E}(W)}{\mathbb{E}(\tau)}\right|\geq\delta\right)\\
&\leq\mathbb{P}\left(\frac{1}{t}\left|\sum_{i=1}^{\lfloor M_t\rfloor}(W_i-\mathbb{E}(W))\right|\geq\delta/2\right)\;+\;\mathbb{P}\left(\mathbb{E}(W)\left|\frac{\lfloor M_t\rfloor}{t}-\frac{1}{\mathbb{E}(\tau)}\right|\geq\delta/2\right)\\
&\leq 2\mathbb{P}\left(\left|\sum_{i=1}^{ct}(W_i-\mathbb{E}(W))\right|>\delta t/4\right)+\mathbb{P}(\lfloor M_t\rfloor>2ct)+\\
&\quad+\;\mathbb{P}\left(\mathbb{E}(W)\left|\frac{\lfloor M_t\rfloor}{t}-\frac{1}{\mathbb{E}(\tau)}\right|\geq\delta/2\right).
\end{aligned}$$

Introducing the Cramer transform of W,

$$\Psi^*(x)=\sup_{\lambda}\{\lambda x-\ln\mathbb{E}(e^{\lambda(W-\mathbb{E}(W))})\}$$

we thus deduce, if Assumption 2.2.4 is fulfilled

$$\limsup_{t\to\infty}\frac{1}{t}\ln\mathbb{P}\left(\left|\frac{Z_t}{t}-\frac{\mathbb{E}(W)}{\mathbb{E}(\tau)}\right|\geq\delta\right)\leq -A(\delta)\,,\tag{2.3.4}$$

where

$$\begin{aligned} A(\delta) &= \min(A_1(\delta), A_2(\delta)) \\ A_1(\delta) &= \min\left(J_\tau\left(\frac{1}{\mathbb{E}(\tau)}+\frac{\delta}{2\mathbb{E}(W)}\right), J_\tau\left(\frac{1}{\mathbb{E}(\tau)}-\frac{\delta}{2\mathbb{E}(W)}\right)\right) \quad (2.3.5) \\ A_2(\delta) &= \sup_{2c>1/\mathbb{E}(\tau)} \min\left(J_\tau(2c)\,,\, c\,\min\left(\psi^*(\delta/4c), \psi^*(-\delta/4c)\right)\right). \quad (2.3.6) \end{aligned}$$

This time however we cannot let c go to infinity so that $A_2(\delta)$ will furnish a worse bound than $\beta_0\delta/2$ (the correcting term in Lemma 2.3.1). So even when $\beta_0 < +\infty$ the previous bound will be worse than the one obtained by combining Lemma 2.3.1 and the LDP for finite valued W.

This kind of approach should be useful in order to get non asymptotic bounds. $\diamondsuit$

2.4 Large deviations for a finite valued W.

From now on we will assume that W takes its values in a set of cardinal n (fixed). However we will keep a superscript n to remember that this is a reduction of the general case, so we shall write W^n. For simplicity we rename the values of W^n as $1, ..., n$. According to Assumption 2.2.4 iii) and the approximation made in subsection 2.3.2 we may assume that

$$\forall j = 1, ..., n\,, \forall t > 0\,, \psi^n(\{\tau > t\} \cap \{w = j\}) = \mathbb{P}(\tau > t, W^n = j) > 0\,.$$

This condition will replace iii) in Assumption 2.2.4 in all the next statements.

Some results are still true when W takes its value in some $\mathbb{W} \subset \mathbb{R}$ which is compact, in particular if W is bounded. We will mention these results in related remarks each time it is possible.

2.4.1 More notations.

As we said we will now closely follow the approach in [52]. Consider the backward recurrence time process $(A_t)_{t\geq 0}$, the forward recurrence time process $(B_t)_{t\geq 0}$ and the process $(C_t^n)_{t\geq 0}$, defined by:

$$A_t = t - S_{M_t}, \qquad B_t = S_{M_t+1} - t, \qquad C_t^n = W^n_{M_t+1}\,.$$

Note that for all time t the random variable $A_t + B_t = S_{M_t+1} - S_{M_t}$ is distributed as τ. In the sequel, we study the associated empirical measure:

$$\mu_t^n := \frac{1}{t}\int_{[0,t)} \delta_{(A_s,B_s,C_s^n)} ds \in \mathcal{M}^1((0,+\infty]^2 \times \{1, ..., n\}).$$

We denote by $\mathbb{P}_t^n$ the law of μ_t^n.

This empirical measure is a probability measure on $(0,+\infty]^2 \times \{1, ..., n\}$. Unlike in the work of Lefevere, Mariani and Zambotti [52] who focused on the specific case where $W = F(\tau)$, we have to consider three coordinates in the empirical measure. Looking at $(0,+\infty]$ allows us to avoid integrability considerations at infinity since A_s and B_s can be as large as we want simultaneously.

Choosing as metric $d(t,t') = |\frac{1}{t} - \frac{1}{t'}|$ on $(0,+\infty]$ and the usual one on $\{1,...,n\}$ we immediately see that $\mathcal{Y} = (0,+\infty]^2 \times \{1,...,n\}$ is a Polish space, so that $\mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$ is also Polish. In addition $(a,b,c) \to \frac{1}{a+b}$ is continuous (of course it is equal to 0 if either a or b equals $+\infty$).

For $\mu \in \mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$, we denote:

$$\mu(1/\tau) = \mu(1/(a+b)) = \int_{(0,+\infty)^2 \times \{1,...,n\}} \frac{1}{a+b}\, \mu(da,db,dc)$$

and for $\pi \in \mathcal{M}^1((0,+\infty] \times \{1,...,n\})$, we denote:

$$\pi(1/\tau) = \int_{(0,+\infty] \times \{1,...,n\}} \frac{1}{\tau}\, \pi(d\tau, dW).$$

Let us define $\Delta_0^n \subset \mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$ as the set of measures such that a and b correspond to a uniform partition of the length τ, i.e. $a = u\tau$ and $b = (1-u)\tau$ where u is a uniform random variable independent of (τ, W):

$$\begin{aligned} \Delta_0^n = \{&\mu_0 \in \mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\}), \\ &\mu_0(da,db,dc) = \int_{[0,1]\times(0,+\infty)\times\{1,...,n\}} \delta_{(u\tau,(1-u)\tau,W)}(da,db,dc)du \otimes \pi(d\tau,dW), \\ &\pi \in \mathcal{M}^1((0,+\infty) \times \{1,...,n\}), \pi(1/\tau) < +\infty\} \end{aligned} \tag{2.4.1}$$

We also define Δ^n contains mixing of an element of Δ_0^n and a measure on $(+\infty,+\infty)\times \{1,\ldots,n\}$ described by the law η:

$$\begin{aligned} \Delta^n = \{&\mu(da,db,dc) = \alpha\mu_0(da,db,dc) + (1-\alpha)\delta_{(+\infty,+\infty)}(da,db) \otimes \eta(dc), \\ &\mu_0 \in \Delta_0^n, \alpha \in [0,1], \eta \in \mathcal{M}^1(\{1,...,n\})\}. \end{aligned} \tag{2.4.2}$$

The following will be used several times in what follows

Lemma 2.4.1. For $M > 0$, the set $\mathcal{X}_M = \{\nu \in \mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})\,,\, \nu(1/(a+b)) \le M\}$ is compact in $\mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$.

Proof. First the set $\{a+b \le \delta\}$ is compact in $(0,+\infty]^2 \times \{1,...,n\}$ for $\delta \ge 0$. It follows that $\mathcal{X}_M$ is a uniformly integrable set, hence is relatively compact (tight). Indeed for $\nu \in \mathcal{X}_M$

$$\nu(\{a+b<\delta\}) \le \delta\, \nu(1/(a+b)) \le \delta\, M\,.$$

Notice that this set is also closed, so that it is compact. Indeed if ν belongs to $\mathcal{X}_M$, $\nu(\frac{1}{a+b+\varepsilon}) \le M$ for all $\varepsilon > 0$. Any weak limit ν of such ν_k's satisfies $\nu(\frac{1}{a+b+\varepsilon}) \le M$ since $(a,b,c) \mapsto 1/(a+b+\varepsilon)$ is bounded and continuous. It remains to apply the monotone convergence theorem. □

Remark 2.4.2. This lemma and its proof are still true if we replace $\{1, ..., n\}$ by a compact set $\mathbb{W}$. $\diamondsuit$

Recall that ψ^n denotes the joint distribution of (τ, W^n), so that its first marginal does not depend on n. We may thus skip the superscript n when dealing with quantities that only depend on this first marginal. To simplify the notation we will denote by ξ the θ_0 in Assumption 2.2.4, i.e.

$$\xi = \sup\{c \in \mathbb{R}, \psi(\mathrm{e}^{c\tau}) < \infty\} \in [0, +\infty]. \tag{2.4.3}$$

For $\pi \in \mathcal{M}^1((0, +\infty] \times \{1, ..., n\})$, satisfying $\pi(1/\tau) \in (0, +\infty)$, we define $\tilde{\pi} \in \mathcal{M}^1((0, +\infty] \times \{1, ..., n\})$ as:

$$\tilde{\pi}(d\tau, dW) = \frac{1}{\pi(1/\tau)} \frac{1}{\tau} \pi(d\tau, dW).$$

$\tilde{\pi}$ has no weight on $\{+\infty\} \times \{1, ..., n\}$.

We also define the functional $I_0^n, I^n : \mathcal{M}^1((0, +\infty]^2 \times \{1, ..., n\}) \to [0, +\infty]$ by:

$$I_0^n(\mu) = \begin{cases} \pi(1/\tau) H(\tilde{\pi}|\psi^n) & \text{if } \mu \in \Delta_0^n \\ \infty & \text{otherwise} \end{cases}, \tag{2.4.4}$$

$$I^n(\mu) = \begin{cases} \alpha\pi(1/\tau) H(\tilde{\pi}|\psi^n) + (1-\alpha)\xi & \text{if } \mu \in \Delta^n \\ \infty & \text{otherwise} \end{cases}, \tag{2.4.5}$$

where H is the relative entropy defined by

$$H(\nu|\mu) = \begin{cases} \displaystyle\int \ln\left(\frac{d\nu}{d\mu}\right) d\mu & \text{if } \nu \text{ is absolutely continuous w.r.t. } \mu \\ \infty & \text{otherwise.} \end{cases}$$

We denote by $\mathcal{C}_b((0, +\infty]^2 \times \{1, ..., n\})$ the set of bounded and continuous functions on $(0, +\infty]^2 \times \{1, ..., n\}$. We introduce a subspace Γ, which will be useful to get a control on the functional I^n.

For a bounded measurable $f : (0, +\infty) \times (0, +\infty) \times \{1, ..., n\} \to \mathbb{R}$, we set :

$$\bar{f}(x, \tau, W) = \int_0^x f(u\tau, (1-u)\tau, W) du \tag{2.4.6}$$

We define:

$$C_{n,f} = \int_{(0,+\infty)\times\{1,\dots,n\}} \mathrm{e}^{\tau \bar{f}(1,\tau,W)} \, \psi^n(d\tau, dW) \,, \tag{2.4.7}$$

and

$$D_{n,f} = \sup_{s>0} \int_{(s,+\infty)} \mathrm{e}^{\tau \bar{f}(s/\tau,\tau,W)} \, \psi^n(d\tau, dW) \,.. \tag{2.4.8}$$

We consider

$$\Gamma := \{f : (0,+\infty]^2 \times \{1,...,n\} \to \mathbb{R}, \text{ bounded, lower semicontinuous,} \\ C_{n,f} < 1, D_{n,f} < +\infty\}. \tag{2.4.9}$$

The reasons for introducing such quantities are detailed in [52] section 1, see in particular subsection 1.5 (and more particularly Remark 1.5) and subsection 1.6.

2.4.2 Large deviations principle for cumulative process when W is finite valued.

We will first prove a LDP for the empirical measure μ_t^n:

Theorem 2.4.3. Let μ_t^n the empirical measure of a cumulative process satisfying Asumptions 2.2.4, and such that W^n takes its values in $\{1,...,n\}$. The family $(\mathbb{P}_t^n)_{t\geq 0}$ of probability distributions of μ_t^n satisfies a large deviations principle with good rate function I^n as $t \to \infty$ with speed t, where I^n is defined in (2.4.5).

Applying the contraction principle we will then deduce

Theorem 2.4.4. Let $(\tau_i, W_i^n)_i$ an i.i.d. sequence of couples of random variables in $(0,+\infty) \times \{1,...,n\}$ following the law ψ^n. Let us define $Z_t^n = \sum_{i=1}^{M_t} W_i^n$ the associated cumulative process. If ψ^n satisfies Asumptions 2.2.4, then the law of Z_t^n/t satisfies a large deviation principle with good rate function J^n (given by (2.2.6)).

2.5 The functional I^n

This section aims at proving that I^n, defined in (2.4.5), is a good rate function i.e proposition 2.5.2 (analogue to Proposition 1.3 in [52]. Three propositions are necessary to prove it: the propositions 2.5.3, 2.5.4 and 2.5.6 which are the analogues of Proposition 2.1 in [52]. In order to prove these three propositions some additional lemmata are necessary.

Remark 2.5.1. If the scheme of proof is close to the one in [52], the proofs of this section need some new ideas. In particular, we will use conditioning by sets like $\{W = j\}$. This induces to use disintegration and uniform controls in j. That is why it is useful to work with a finite valued W. In addition the compactness of $\mathcal{M}_1(\{1,...,n\})$ will help to control the third coordinate (which does not exist in [52]). Except two auxiliary lemmata, we decided to give self contained proofs. $\diamondsuit$

Proposition 2.5.2. The function I^n is a good rate function. Moreover, I^n is the lower semicontinuous envelope of I_0^n.

For the proof we need the next three propositions. We need another notation: if ψ_k is a sequence in $\mathcal{M}^1((0,+\infty) \times \{1,...,n\})$, ξ_k and I_k^n are defined as in (2.4.3) and (2.4.5) respectively, with ψ^n replaced by ψ_k.

Proposition 2.5.3. Let ψ_k be a sequence in $\mathcal{M}^1((0,+\infty)\times\{1,...,n\})$. Assume that $\psi_k \rightharpoonup \psi^n$, and $\xi_k \to \xi$ as $k\to+\infty$. Then any sequence $(\mu_k)_k$ in $\mathcal{M}^1((0,+\infty]^2\times\{1,...,n\})$ such that $\limsup_{k\to\infty} I^n_k(\mu_k)<\infty$ is tight and thus, relatively compact in $\mathcal{M}^1((0,+\infty]^2\times\{1,...,n\})$. In particular, for k large enough, such a sequence is in Δ^n. If we consider π_k and α_k the associated quantities, we have:

$$\limsup_{k\to\infty}\alpha_k\pi_k(1/\tau)<+\infty. \tag{2.5.1}$$

Proposition 2.5.4. Let ψ_k be a sequence in $\mathcal{M}^1((0,+\infty)\times\{1,...,n\})$. Assume that $\psi_k \rightharpoonup \psi^n$, and $\xi_k \to \xi$ as $k\to+\infty$. Then for any μ and any sequence $(\mu_k)_k$ in $\mathcal{M}^1((0,+\infty]^2\times\{1,...,n\})$, such that $\mu_k\rightharpoonup\mu$, we have $\liminf_{k\to\infty} I^n_k(\mu_k)\geq I^n(\mu)$.

Remark 2.5.5. Both previous propositions are still true when replacing $\{1,...,n\}$ by $\mathbb{W}$ a compact subset of $\mathbb{R}$ as it will be clear looking at their proof. $\diamondsuit$

Proposition 2.5.6. Let ψ_k be a sequence in $\mathcal{M}^1((0,+\infty)\times\{1,...,n\})$. Assume that $\psi_k \rightharpoonup \psi^n$, and $\xi_k \to \xi$ as $k\to+\infty$. Then for any μ in $\mathcal{M}^1((0,+\infty]^2\times\{1,...,n\})$ with $I^n(\mu)<\infty$, there exists a sequence $(\mu_k)_k$ such that $\mu_k\rightharpoonup\mu$, $\mu_k\in\Delta^n_0$ for all k and $\limsup_{k\to\infty} I^n_k(\mu_k)\leq I^n(\mu)$.
Moreover, we have $\mu_k\left(\frac{1}{a+b}\right)\underset{k\to\infty}{\longrightarrow}\mu\left(\frac{1}{a+b}\right)$.

For this last proposition we need to work with W taking values in a finite set.

Now we can prove the main Proposition of this section.

Proof of Proposition 2.5.2. We want to prove that I^n is a good rate function and is the lower semicontinuous envelope of I^n_0.
We apply Proposition 2.5.3 with $\psi_k=\psi^n$: for all $(\mu_k)_k$ sequence of $\mathcal{M}^1((0,+\infty]^2\times\{1,...,n\})$, if $\limsup_{k\to\infty} I^n(\mu_k)<+\infty$, then the family $(\mu_k)_k$ is tight in $\mathcal{M}^1((0,+\infty]^2\times\{1,...,n\})$. We can deduce that I^n has relatively compact sublevel sets (and is coercive).
Thanks to Proposition 2.5.4 we have, for any μ and any sequence μ_k in $\mathcal{M}^1((0,+\infty]^2\times\{1,...,n\})$ such that $\mu_k\rightharpoonup\mu$, $\liminf_{k\to\infty} I^n(\mu_k)\geq I^n(\mu)$. Thus I^n has closed sublevel sets. It also means that I^n is lower semicontinuous.
I^n has relatively compact and closed sublevel sets, so I^n is a good rate function.

We already know that $I^n\leq I^n_0$ since $\Delta^n_0\subset\Delta^n$, and I^n is lower semicontinuous, so I^n is smaller than or equal to the lower semicontinuous envelope of I^n_0.
Thanks to proposition 2.5.6, for any $\mu\in\mathcal{M}^1((0,+\infty]^2\times\{1,...,n\})$ with $I^n(\mu)<\infty$, there exists $(\mu_k)_k$ a sequence in Δ^n_0 such that $\mu_k\rightharpoonup\mu$ and $\limsup_{k\to\infty} I^n(\mu_k)\leq I^n(\mu)$. In fact, $\limsup_{k\to\infty} I^n_0(\mu_k)\leq I^n(\mu)$. So I^n is greater or equal to the lower semicontinuous envelope of I^n_0 and finally I^n is the lower semicontinuous envelope of I^n_0. $\square$

We turn to the proof of the three auxiliary propositions.

Proof of proposition 2.5.3. Choose a sequence $(\mu_k)_k$ in $\mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$ such that $\limsup_{k\to\infty} I_k^n(\mu_k) < \infty$. We want to prove that this sequence is tight and thus, relatively compact in $\mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$.

Since $\limsup_{k\to\infty} I_k^n(\mu_k) < \infty$, for k large enough, $\mu_k \in \Delta^n$. Then, there exist $\alpha_k \in [0,1]$, $\pi_k \in \mathcal{M}^1((0,+\infty) \times \{1,...,n\})$ with $\pi_k(1/\tau) < +\infty$ and $\eta_k \in \mathcal{M}^1(\{1,...,n\})$.

First, we prove equation (2.5.1):

$$\limsup_{k\to\infty} \alpha_k \pi_k(1/\tau) < +\infty.$$

We have $I_k^n(\mu_k) = \alpha_k \pi_k(1/\tau) H(\tilde{\pi}_k|\psi_k) + (1-\alpha)\xi_k \geq \alpha_k \pi_k(1/\tau) H(\tilde{\pi}_k|\psi_k)$. Therefore

$$\alpha_k \pi_k(1/\tau) \leq \frac{I_k^n(\mu_k)}{H(\tilde{\pi}_k|\psi_k)} \text{ (we consider } H(\tilde{\pi}_k|\psi_k) \neq 0).$$

Moreover

$$\alpha_k \pi_k(1/\tau) = \alpha_k \frac{1}{\tilde{\pi}_k(\tau)}.$$

Then

$$\alpha_k \pi_k(1/\tau) \leq \frac{I_k^n(\mu_k)}{H(\tilde{\pi}_k|\psi_k)} \wedge \frac{\alpha_k}{\tilde{\pi}_k(\tau)}$$

$$\text{and } \limsup_{k\to\infty} \alpha_k \pi_k(1/\tau) \leq \limsup_{k\to\infty} \frac{I_k^n(\mu_k)}{H(\tilde{\pi}_k|\psi_k)} \wedge \frac{\alpha_k}{\tilde{\pi}_k(\tau)}.$$

$\alpha_k \in [0,1]$ and $\limsup_{k\to\infty} I_k^n(\mu_k) < +\infty$, so there exists a constant $C < \infty$ such that $\forall k, \alpha_k \leq C$ and $I_k^n(\mu_k) \leq C$. Then

$$\begin{aligned}
\limsup_{k\to\infty} \alpha_k \pi_k(1/\tau) &\leq \limsup_{k\to\infty} \frac{C}{H(\tilde{\pi}_k|\psi_k)} \wedge \frac{C}{\tilde{\pi}_k(\tau)} \\
&\leq C \limsup_{k\to\infty} \frac{1}{H(\tilde{\pi}_k|\psi_k)} \wedge \frac{1}{\tilde{\pi}_k(\tau)} \\
&\leq C \limsup_{k\to\infty} \frac{1}{H(\tilde{\pi}_k|\psi_k) \vee \tilde{\pi}_k(\tau)} \\
&\leq \frac{C}{\liminf_{k\to\infty} H(\tilde{\pi}_k|\psi_k) \vee \tilde{\pi}_k(\tau)}.
\end{aligned}$$

If there exists a subsequence k_j such that $H(\tilde{\pi}_{k_j}|\psi_{k_j}) \to 0$, then $\lim \tilde{\pi}_{k_j} = \lim \psi_{k_j} = \psi^n$. Thus, since $[0,1]$ and $\mathcal{M}^1(\{1,...,n\})$ are compact, there exist subsequences of α_k and η_k which converge. Eventually, there exists a subsequence of μ_k which converges, so this sequence is relatively compact. In particular, (2.5.1) is true because $\liminf_{j\to\infty} \tilde{\pi}_{k_j}(\tau) = \psi^n(\tau) > 0$.

Else, $\liminf_{k\to\infty} H(\tilde{\pi}_k|\psi_k) > 0$ and $\limsup_{k\to\infty} \alpha_k \pi_k(1/\tau) < \infty$. Then (2.5.1) is true.

Moreover, for M large enough and k large enough, $\mu_k(1/(a+b)) = \alpha_k\pi_k(1/\tau) \le M$. So $\mu_k \in \mathcal{X}_M$ which is a compact set by Lemma 2.4.1. Then, this sequence is relatively compact. In both cases, the sequence is relatively compact and (2.5.1) is true. □

Remark 2.5.7. As already said we only used the compactness of $\mathcal{M}^1(\mathbb{W})$ and lemma 2.4.1 so that the above proof immediately extends to $\mathbb{W}$ compact. ♢

Proof of proposition 2.5.4. Let $\mu \in \mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$ and let $(\mu_k)_k$ in $\mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$, such that $\mu_k \rightharpoonup \mu$. We want to prove that $\liminf_{k\to\infty} I_k^n(\mu_k) \ge I^n(\mu)$.

Since $\mu_k \rightharpoonup \mu$, we may replace μ_k by subsequences again denoted μ_k in what follows. We assume $\sup_k I_k^n(\mu_k) < \infty$ (otherwise, if $\liminf_{k\to\infty} I_k^n(\mu_k) = +\infty$, $+\infty \ge I^n(\mu)$). Then $\mu_k \in \Delta^n$ and we denote by α_k, π_k, η_k and $\mu_{0,k}$ the corresponding quantities, with $\alpha_k \in [0,1]$ and $\pi_k(1/\tau) < \infty$.

<u>First, if $\limsup_{k\to\infty} \alpha_k = 0$,</u> we may assume taking a subsequence that $\lim \alpha_k = 0$. So $\mu = \delta_{(+\infty,+\infty)} \otimes (\lim_k \eta_k)$. Since $\mathcal{M}^1(\{1,...,n\})$ is compact, $(\lim_k \eta_k) \in \mathcal{M}^1(\{1,...,n\})$. Therefore

$$\liminf_{k\to\infty} I_k^n(\mu_k) = \liminf_{k\to\infty} \alpha_k\pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k)+(1-\alpha_k)\xi_k \ge \liminf_{k\to\infty}(1-\alpha_k)\xi_k = \xi = I^n(\mu).$$

<u>Secondly, we have $\limsup_{k\to\infty} \alpha_k = \bar{\alpha} > 0$.</u> Again we may assume that $\lim \alpha_k = \bar{\alpha}$.

We begin by studying the sequence μ_k and π_k to have some information about $\liminf_{k\to\infty} I_k^n(\mu_k)$. Then, we prove that μ is in Δ^n, and finally we work on $I^n(\mu)$.

Since $\sup_k I_k^n(\mu_k) < \infty$, we can apply the proposition 2.5.3 and in particular the equation (2.5.1): $\limsup_{k\to\infty} \alpha_k\pi_k(1/\tau) < \infty$. Since $\bar{\alpha} > 0$, we have: $\limsup_{k\to\infty} \pi_k(1/\tau) < \infty$. For k large enough and M large enough, we have $\pi_k(1/\tau) \le M$.

As

$$\left\{\nu \in \mathcal{M}^1((0,+\infty] \times \{1,...,n\}), \nu\left(\frac{1}{\tau}\right) \le r\right\}$$

is tight for all $r > 0$, there exists a subsequence of π_k which converges in $\mathcal{M}^1((0,+\infty] \times \{1,...,n\})$. We can write the limit of π_k (or its subsequence) as $\beta\pi + (1-\beta)\delta_{(+\infty)} \otimes \eta_0$, for some $\beta \in [0,1]$ and $\eta_0 \in \mathcal{M}^1(\{1,...,n\})$.

If $\beta > 0$, $\pi(1/\tau) \le \frac{1}{\beta}\limsup_{k\to\infty} \pi_k(1/\tau) < \infty$. If $\beta = 0$ we choose an arbitrary π such that $\pi(1/\tau) < \infty$.

Now, we prove that μ is in Δ^n, where μ is defined by:

$$\begin{aligned}
\mu &= \lim_k \mu_k \\
&= \lim_k(\alpha_k\mu_{0,k} + (1-\alpha_k)\delta_{(+\infty,+\infty)} \otimes \eta_k) \\
&= \bar{\alpha}\lim_k \mu_{0,k} + (1-\bar{\alpha})\delta_{(+\infty,+\infty)} \otimes (\lim_k \eta_k).
\end{aligned}$$

It holds

$$\mu_{0,k} = \int_{[0,1]\times(0,+\infty)\times\{1,...,n\}} \delta_{(u\tau,(1-u)\tau,W)} du \otimes \pi_k(d\tau, dW)$$
$$\xrightarrow[k\to\infty]{} \beta \int_{[0,1]\times(0,+\infty)\times\{1,...,n\}} \delta_{(u\tau,(1-u)\tau,W)} du \otimes \pi(d\tau, dW) + (1-\beta)\delta_{(+\infty,+\infty)} \otimes \eta_0.$$

Let μ_0 and η be defined as:

$$\mu_0 := \int_{[0,1]\times(0,+\infty)\times\{1,...,n\}} \delta_{(u\tau,(1-u)\tau,W)} du \otimes \pi(d\tau, dW)$$
$$\eta := \frac{\alpha(1-\beta)}{1-\bar{\alpha}\beta}\eta_0 + \frac{1-\bar{\alpha}}{1-\bar{\alpha}\beta}(\lim_k \eta_k).$$

We get

$$\begin{aligned}\mu &= \bar{\alpha}\beta\mu_0 + \bar{\alpha}(1-\beta)\delta_{(+\infty,+\infty)} \otimes \eta_0 + (1-\bar{\alpha})\delta_{(+\infty,+\infty)} \otimes (\lim_k \eta_k) \\ &= \bar{\alpha}\beta\mu_0 + (1-\bar{\alpha}\beta)\delta_{(+\infty,+\infty)} \otimes \eta,\end{aligned}$$

and in particular $\mu \in \Delta^n$ with $\alpha = \beta\bar{\alpha}$.

Eventually

$$\begin{aligned}I^n(\mu) &= \beta\bar{\alpha}\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1-\beta\bar{\alpha})\xi \\ &= \bar{\alpha}\left[\beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1-\beta)\xi\right] + (1-\bar{\alpha})\xi \\ &= \bar{\alpha}\pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k) + (1-\bar{\alpha})\xi \\ &\quad + \bar{\alpha}\left[\beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1-\beta)\xi - \pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k)\right] \\ &\le \liminf_{k\to\infty} I^n_k(\mu_k) + \bar{\alpha}\left[\beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1-\beta)\xi - \pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k)\right].\end{aligned}$$

In particular, we can apply Lemma 2.4 from [52] (the proof can be directly adapted in our case - see 2.A.3), since the hypotheses therein are satisfied:

Lemma 2.5.8. Let $\pi_k \in \mathcal{M}^1((0,+\infty)\times\{1,...,n\})$ be such that $\pi_k(1/\tau) < \infty$ such that

$$\lim_k \pi_k(d\tau, dW) = \beta\pi(d\tau, dW) + (1-\beta)\delta_{(+\infty)}(d\tau) \otimes \eta_0(dW) \tag{2.5.2}$$

for some $\beta \in [0,1]$, $\pi \in \mathcal{M}^1((0,+\infty)\times\{1,...,n\})$ such that $\pi(1/\tau) < \infty$ and $\eta_0 \in \mathcal{M}^1(\{1,...,n\})$. Then

$$\liminf_{k\to\infty} \pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k) \ge \beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1-\beta)\xi.$$

Eventually

$$\begin{aligned}I^n(\mu) &\le \liminf_{k\to\infty} I^n_k(\mu_k) + \bar{\alpha}\underbrace{\left[\beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1-\beta)\xi - \pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k)\right]}_{<0 \text{ by Lemma } 2.5.8} \\ &\le \liminf_{k\to\infty} I^n_k(\mu_k).\end{aligned}$$

□

Remark 2.5.9. Once again we only used the fact that $\{1,...,n\}$ is compact so that we may replace it by any compact $\mathbb{W}$. $\diamondsuit$

Proof of proposition 2.5.6. Let μ in $\mathcal{M}^1((0,+\infty]^2\times\{1,...,n\})$ with $I^n(\mu)<\infty$. We want to prove that there exists a sequence (μ_k) such that $\mu_k \rightharpoonup \mu$, $\mu_k\in\Delta_0^n$ for all k and $\limsup_{k\to\infty} I_k^n(\mu_k)\le I^n(\mu)$.
Since $I^n(\mu)<\infty$, $\mu\in\Delta^n$. Let α, π and η corresponding to μ:

$$\begin{aligned}\mu(da,db,dc) =&\alpha\int_{[0,1]\times(0,+\infty)\times\{1,...,n\}}\delta_{(u\tau,(1-u)\tau,W)}(da,db,dc)du\otimes\pi(d\tau,dW)\\ &+(1-\alpha)\delta_{(+\infty,+\infty)}(da,db)\otimes\eta(dc).\end{aligned}$$

The aim of this (technical) proof is to construct a sequence of laws π_k, depending on ψ_k, which satisfies a condition on its limit (the condition (2.5.6) described below) and a condition on its entropy with respect to ψ_k (the condition (2.5.7) described below). Then, we will construct a sequence of measure μ_k which will satisfy the wished conditions. The next paragraphs give the details.

We denote $\psi^n(d\tau)$ the marginal law of τ in ψ^n, and $\psi_k(d\tau)$ the corresponding law associated to ψ_k to simplify notations. In particular, $\psi^n(d\tau|W=j)$ is the marginal law of τ given $W=j$. We write q_j the weight of j for η: $q_j=\eta(\{W=j\})$.

Fix $\rho>0, L>M>1$ such that $\mathbb{P}_{\psi^n}(\tau=1/M)=\mathbb{P}_{\psi^n}(\tau=M)=\mathbb{P}_{\psi^n}(\tau=L)=0$. Then there exist $N\in\mathbb{N}$ and $1/M=T_1<T_2<...<T_N=M$ such that $T_{i+1}-T_i\le\rho$ and $\mathbb{P}_{\psi^n}(\tau=T_i)=0$.
Here of course N and T_i depend on M and ρ. We also use the shorthand notation $A_i=[T_i,T_{i+1})$ and $A=\cup_{i=1}^N A_i$ in this proof. Then for $L>M$ define $\pi_{0,k}^{\rho,L,M}(d\tau,dW)$, $\eta_k^{L,M}(d\tau,dW)$ and $\pi_k^{\rho,M,L}(d\tau,dW)\in\mathcal{M}^1((0,+\infty)\times\{1,...,n\})$ as:

$$\pi_{0,k}^{\rho,M}(d\tau,dW)=\frac{1}{\beta_k^{\rho,M}}\sum_{i=1}^N\sum_{j=1}^n\frac{\tilde\pi(\tau\in A_i,W=j)}{\tilde\pi(A)}\tau\psi_k(d\tau,dW|\tau\in A_i,W=j),\tag{2.5.3}$$

$$\eta_k^{L,M}(d\tau,dW)=\sum_{j=1}^n\frac{1}{\psi_k(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\tau\mathbb{1}_{\tau\in[M,L)}q_j\psi_k(d\tau|W=j)\otimes\delta_j(dW),\tag{2.5.4}$$

$$\pi_k^{\rho,L,M}(d\tau,dW)=\alpha\pi_{0,k}^{\rho,M}(d\tau,dW)+(1-\alpha)\eta_k^{L,M}(d\tau,dW),\tag{2.5.5}$$

where $\beta_k^{\rho,M}$ is the normalizing constant such that $\pi_{0,k}^{\rho,M}$ is a probability measure.

Some q_j's can be equal to 0. In these cases, we consider in fact the sum on j such that the q_j aren't zero: all the following calculation are then true. In (2.5.4), and in the following calculus, we consider the sum on i, j, such that the denominators are not null. To simplify the notation, we consider in the following calculus that $\eta_k^{L,M}$

is the sum for $j \in \{1, ..., n\}$.

The above definition makes sense if $L > M$ is large enough, and k is large enough depending on L and M (k will be sent to $+\infty$ before L, and L before M), and there is no problem with the conditioning. As $\psi^n(\tau \in \partial A_i) = 0$, if $\psi_k(A_i \times \{j\}) = 0$ for each k large enough, then $\psi(A_i \times \{j\}) = 0$. Since $I^n(\mu) < \infty$, then $H(\tilde{\pi}|\psi^n) < \infty$ and $\tilde{\pi}(A_i \times \{j\}) = 0$. The term in (i, j) in (2.5.3) would be considered equal to 0. Similarly if $\psi_k([M, L) \times \{n\}) = 0$ then the term in j in (2.5.4) vanishes. If each terms of one sum is zero, then α is equal to 0 or 1, by the same arguments on the relative entropy.

We want to prove:

$$\lim_{M\to\infty} \lim_{L\to\infty} \lim_{\rho\to 0} \lim_{k\to\infty} \pi_k^{\rho,L,M} = \alpha\pi + (1-\alpha)\delta_{(+\infty)} \otimes \eta, \tag{2.5.6}$$

$$\liminf_{M\to\infty} \liminf_{L\to\infty} \liminf_{\rho\to 0} \limsup_{k\to\infty} \pi_k^{\rho,L,M}(1/\tau)H(\tilde{\pi}_k^{\rho,L,M}|\psi_k) \leq I^n(\mu), \tag{2.5.7}$$

with $I^n(\mu) = \alpha\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1-\alpha)\xi$, where the limits in M and L are understood to run over M and L satisfying the above conditions.

Once (2.5.6) and (2.5.7) are proved, we can consider sequences $\rho_k \to 0$, $L_k \to \infty$, $M_k \to \infty$ (such that ρ_k, L_k and M_k satisfy the above conditions), we can define $\pi_k = \pi_k^{\rho_k, L_k, M_k}$ which satisfy: $\pi_k \rightharpoonup \alpha\pi + (1-\alpha)\delta_{(+\infty)} \otimes \eta$ and

$$\limsup_{k\to\infty} \pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k) \leq I^n(\mu).$$

Then μ_k defined by:

$$\mu_k(da, db, dc) := \int_{[0,1]\times(0,+\infty)\times\{1,...,n\}} \delta_{(u\tau,(1-u)\tau,W)}(da, db, dc)du \otimes \pi_k(d\tau, dW)$$

satisfies the proposition.

First, we prove the convergence (2.5.6) step by step. First, we consider $\tilde{\pi}_{0,k}^{\rho,M}$. The normalisation constant $\beta_k^{\rho,M}$ satisfies:

$$\begin{aligned}
\beta_k^{\rho,M} &= \int_{(0,+\infty)\times\{1,...,n\}} \sum_{i=1}^{N} \sum_{j=1}^{n} \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \tau\psi_k(d\tau, dW|\tau \in A_i, W = j) \\
&= \sum_{i=1}^{N} \sum_{j=1}^{n} \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \int_{A_i\times\{j\}} \tau\psi_k(d\tau, dW|\tau \in A_i, W = j) \\
&= \sum_{i=1}^{N} \sum_{j=1}^{n} \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \psi_k(\tau|\tau \in A_i, W = j)
\end{aligned}$$

Since $\psi_k \underset{k\to\infty}{\longrightarrow} \psi$, then $\forall i \leq N, \forall j \leq n, \psi_k(\tau \in A_i, W = j) \underset{k\to\infty}{\longrightarrow} \psi^n(\tau \in A_i, W = j)$.
By bounded convergence (because $\tau \in A_i \Rightarrow \tau \leq T_{i+1}$), $\forall j \leq n, \forall f \in \mathcal{C}_b((0,+\infty) \times \{1,...,n\})$,

$$\int_{A_i\times\{j\}} \tau f(\tau, W)\psi_k(d\tau, dW) \underset{k\to\infty}{\longrightarrow} \int_{A_i\times\{j\}} \tau f(\tau, W)\psi^n(d\tau, dW).$$

So:

$$\beta_k^{\rho,M} \to \beta^{\rho,M} := \sum_{i=1}^{N}\sum_{j=1}^{n} \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)}\psi^n(\tau|\tau \in A_i, W = j),$$

and

$$\pi_{0,k}^{\rho,M}(d\tau, dW) \to \pi_0^{\rho,M}(d\tau, dW),$$

$$\text{with } \pi_0^{\rho,M}(d\tau, dW) := \frac{1}{\beta^{\rho,M}} \sum_{i=1}^{N}\sum_{j=1}^{n} \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)}\tau\psi^n(d\tau, dW|\tau \in A_i, W = j).$$

Let $f \in \mathcal{C}_b((0,+\infty) \times \{1,...,n\})$. On $A = [1/M, M)$, for each $j \leq n$, $\tau \mapsto \tau f(\tau, j)$ is uniformly continuous, so there exists a modulus of continuity ω_f, such that $\forall (x, y) \in A, \forall j \in \{1,...,n\}, |xf(x, j) - yf(y, j)| \leq \omega_f(|x - y|)$. In fact, ω_f is the maximum of the n modulus of continuity for functions $\tau \mapsto \tau f(\tau, j)$.
For all $\rho > 0$, $(T_i)_i$ defined as before,

$$\begin{aligned}\int_{A\times\{1,...,n\}} \frac{f(\tau, W)\tau}{\tilde{\pi}(\tau \in A)}\tilde{\pi}(d\tau, dW) &= \sum_{j=1}^{n} \int_{A\times\{j\}} \frac{f(\tau, j)\tau}{\tilde{\pi}(\tau \in A)}\tilde{\pi}(d\tau, dW) \\ &= \sum_{j=1}^{n}\sum_{i=1}^{N} \int_{A_i\times\{j\}} \frac{f(\tau, j)\tau}{\tilde{\pi}(\tau \in A)}\tilde{\pi}(d\tau, dW).\end{aligned}$$

Then:

$$\begin{aligned}\sum_{j=1}^{n}\sum_{i=1}^{N} \int_{A_i\times\{j\}} \frac{\min_{\tau\in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)}\tilde{\pi}(d\tau, dW) &\leq \int_{A\times\{1,...,n\}} \frac{\tau f(\tau, W)}{\tilde{\pi}(\tau \in A)}\tilde{\pi}(d\tau, dW) \\ &\leq \sum_{j=1}^{n}\sum_{i=1}^{N} \int_{A_i\times\{j\}} \frac{\max_{\tau\in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)}\tilde{\pi}(d\tau, dW)\end{aligned}$$

$$\begin{aligned}\sum_{j=1}^{n}\sum_{i=1}^{N} \frac{\min_{\tau\in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)}\tilde{\pi}(\tau \in A_i, W = j) &\leq \int_{A\times\{1,...,n\}} \frac{\tau f(\tau, W)}{\tilde{\pi}(\tau \in A)}\tilde{\pi}(d\tau, dW) \\ &\leq \sum_{j=1}^{n}\sum_{i=1}^{N} \frac{\max_{\tau\in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)}\tilde{\pi}(\tau \in A_i, W = j)\end{aligned}$$

Let us remark that

$$\sum_{j=1}^{n}\sum_{i=1}^{N}\frac{\max_{\tau\in A_i}(\tau f(\tau,j))}{\tilde\pi(\tau\in A)}\tilde\pi(\tau\in A_i,W=j)-\sum_{j=1}^{n}\sum_{i=1}^{N}\frac{\min_{\tau\in A_i}(\tau f(\tau,j))}{\tilde\pi(\tau\in A)}\tilde\pi(\tau\in A_i,W=j)$$
$$=\frac{1}{\tilde\pi(\tau\in A)}\sum_{j=1}^{n}\sum_{i=1}^{N}\tilde\pi(\tau\in A_i,W=j)\left(\max_{\tau\in A_i}(\tau f(\tau,j))-\min_{\tau\in A_i}(\tau f(\tau,j)\right)$$
$$\leq\frac{1}{\tilde\pi(A)}\sum_{j=1}^{n}\sum_{i=1}^{N}\tilde\pi(\tau\in A_i,W=j)\omega_f(\rho)=\omega_f(\rho)\underset{\rho\to0}{\longrightarrow}0.$$

Then

$$\sum_{j=1}^{n}\sum_{i=1}^{N}\frac{\min_{\tau\in A_i}(\tau f(\tau,j))}{\tilde\pi(\tau\in A)}\tilde\pi(\tau\in A_i,W=j)\underset{\rho\to0}{\longrightarrow}\int_{A\times\{1,\dots,n\}}\frac{\tau f(\tau,W)}{\tilde\pi(A)}\tilde\pi(d\tau,dW)$$

and

$$\sum_{j=1}^{n}\sum_{i=1}^{N}\frac{\max_{\tau\in A_i}(\tau f(\tau,j))}{\tilde\pi(\tau\in A)}\tilde\pi(\tau\in A_i,W=j)\underset{\rho\to0}{\longrightarrow}\int_{A\times\{1,\dots,n\}}\frac{\tau f(\tau,W)}{\tilde\pi(A)}\tilde\pi(d\tau,dW).$$

Then, by studying $\beta^{\rho,M}\pi_0^{\rho,M}(f)$:

$$\sum_{i=1}^{N}\sum_{j=1}^{n}\frac{\tilde\pi(\tau\in A_i,W=j)}{\tilde\pi(A)}\min_{\tau\in A_i}(\tau f(\tau,j))$$
$$\leq\sum_{i=1}^{N}\sum_{j=1}^{n}\frac{\tilde\pi(\tau\in A_i,W=j)}{\tilde\pi(A)}\int\tau f(\tau,W)\psi^n(d\tau,dW|\tau\in A_i,W=j)$$
$$\leq\sum_{i=1}^{N}\sum_{j=1}^{n}\frac{\tilde\pi(\tau\in A_i,W=j)}{\tilde\pi(A)}\max_{\tau\in A_i}(\tau f(\tau,j)).$$

So:

$$\forall f\in\mathcal{C}_b((0,+\infty)\times\{1,...,n\}),\beta^{\rho,M}\pi_0^{\rho,M}(f)\underset{\rho\to0}{\longrightarrow}\int_{A\times\{1,\dots,n\}}\frac{\tau f(\tau,W)}{\tilde\pi(A)}\tilde\pi(d\tau,dW).\tag{2.5.8}$$

In particular, for $f=\mathbb{1}$, $\pi_0^{\rho,M}(\mathbb{1})=1$, and:

$$\beta^{\rho,M}\underset{\rho\to0}{\longrightarrow}\beta^M:=\int_{A\times\{1,\dots,n\}}\frac{\tau}{\tilde\pi(A)}\tilde\pi(d\tau,dW)=\tilde\pi(\tau|\tau\in A).$$

Then

$$\pi_0^{\rho,M}(d\tau,dW)\rightharpoonup\pi_0^M(d\tau,dW):=\frac{1}{\beta^M}\tau\tilde\pi(d\tau,dW|\tau\in A).$$

When $M\to\infty$, $\pi(\tau\in A)=\pi(\tau\in[\frac{1}{M},M))\to1$. So: $\beta^M\underset{M\to\infty}{\longrightarrow}\tilde\pi(\tau)$ and:

$$\pi_0^M(d\tau,dW)\rightharpoonup\tau\tilde\pi(d\tau,dW)/\tilde\pi(\tau)=\pi(d\tau,dW).$$

Now, we study $\eta_k^{L,M}$:

$$\eta_k^{L,M}(d\tau, dW) = \sum_{j=1}^{n} \frac{1}{\psi_k(\tau \mathbb{1}_{\tau\in[M,L)}|W=j)} \tau \mathbb{1}_{\tau\in[M,L)} q_j \psi_k(d\tau|W=j) \otimes \delta_j(dW).$$

We have $\psi_k(\tau \mathbb{1}_{\tau\in[M,L)}|W=j) \underset{k\to\infty}{\longrightarrow} \psi^n(\tau \mathbb{1}_{\tau\in[M,L)}|W=j)$, so:

$$\eta_k^{L,M}(d\tau, dW) \to \eta^{L,M}(d\tau, dW),$$

with

$$\eta^{L,M}(d\tau, dW) := \sum_{j=1}^{n} \frac{1}{\psi^n(\tau \mathbb{1}_{\tau\in[M,L)}|W=j)} \tau \mathbb{1}_{\tau\in[M,L)} q_j \psi^n(d\tau|W=j) \otimes \delta_j(dW).$$

When $L \to \infty$,

$$\eta^{L,M}(d\tau, dW) \to \eta^M(d\tau, dW),$$

with

$$\eta^M(d\tau, dW) := \sum_{j=1}^{n} \frac{1}{\psi^n(\tau \mathbb{1}_{\tau\in[M,+\infty)}|W=j)} \tau \mathbb{1}_{\tau\in[M,+\infty)} q_j \psi^n(d\tau|W=j) \otimes \delta_j(dW).$$

For $g \in \mathcal{C}_b((0,+\infty) \times \{1, ..., n\})$ with a compact support,

$$\eta^M(g) = \sum_{j=1}^{n} \int_{[M,\infty)} \frac{1}{\psi^n(\tau \mathbb{1}_{\tau\in[M,+\infty)}|W=j)} \tau g(\tau, j) q_j \psi^n(d\tau|W=j).$$

Thus, for M large enough, $\eta^M(g) = 0$.

Moreover, for $i \in \{1, ..., n\}$,

$$\begin{aligned}
\eta^M(\mathbb{1}_{W=i}) &= \int_{[M,+\infty)} \frac{1}{\psi^n(\tau \mathbb{1}_{\tau\in[M,+\infty)}|W=i)} \tau q_i \psi^n(d\tau|W=i) \\
&= q_i \frac{\psi^n(\tau \mathbb{1}_{\tau\in[M,+\infty)}|W=i)}{\psi^n(\tau \mathbb{1}_{\tau\in[M,+\infty)}|W=i)} = q_i.
\end{aligned}$$

Then: $\eta^M(d\tau, dW) \rightharpoonup \delta_{(+\infty)}(d\tau) \otimes \eta(dW)$.

Now we deduce the convergence of $\pi_k^{\rho,L,M}$:

$$\lim_{M\to\infty} \lim_{L\to\infty} \lim_{\rho\to 0} \lim_{k\to\infty} \pi_k^{\rho,L,M} = \alpha\pi + (1-\alpha)\delta_{(+\infty)} \otimes \eta.$$

(2.5.6) is then proved.

Now we prove the equation (2.5.7). We define $\tilde{\pi}_k^{\rho,L,M}$, $\tilde{\pi}_{0,k}^{\rho,M}$ and $\tilde{\eta}_k^{L,M}$ by:

$$\tilde{\pi}_k^{\rho,L,M}(d\tau,dW) := \frac{1}{\pi_k^{\rho,L,M}(1/\tau)}\frac{1}{\tau}\pi_k^{\rho,L,M}(d\tau,dW), \tag{2.5.9}$$

$$\tilde{\pi}_{0,k}^{\rho,M}(d\tau,dW) := \frac{1}{\pi_{0,k}^{\rho,M}(1/\tau)}\frac{1}{\tau}\pi_{0,k}^{\rho,M}(d\tau,dW), \tag{2.5.10}$$

$$\tilde{\eta}_k^{L,M}(d\tau,dW) := \frac{1}{\eta_k^{L,M}(1/\tau)}\frac{1}{\tau}\eta_k^{L,M}(d\tau,dW). \tag{2.5.11}$$

In particular:

$$\begin{aligned}\eta_k^{L,M}(1/\tau) &= \sum_{j=1}^n \int \frac{1}{\psi_k(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\frac{\tau}{\tau}\mathbb{1}_{\tau\in[M,L)}q_j\psi_k(d\tau|W=j)\otimes\delta_j(dW)\\ &= \sum_{j=1}^n q_j\frac{\psi_k(\mathbb{1}_{\tau\in[M,L)}|W=j)}{\psi_k(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\left(=\sum_{j=1}^n q_j\frac{1}{\psi_k(\tau|\tau\in[M,L),W=j)}\right)\end{aligned}$$

$$\begin{aligned}\tilde{\eta}_k^{L,M}(d\tau,dW) &= \frac{1}{\eta_k^{L,M}(1/\tau)}\frac{1}{\tau}\sum_{j=1}^n\frac{1}{\psi_k(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\tau\mathbb{1}_{\tau\in[M,L)}q_j\psi_k(d\tau|W=j)\otimes\delta_j(dW)\\ &= \frac{1}{\eta_k^{L,M}(1/\tau)}\sum_{j=1}^n\frac{1}{\psi_k(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\mathbb{1}_{\tau\in[M,L)}q_j\psi_k(d\tau|W=j)\otimes\delta_j(dW).\end{aligned}$$

Using Lemma 2.3 from [52] (the proof of which can be easily adapted in this situation - see Lemma 2.A.2) , $\pi\mapsto\pi(1/\tau)H(\tilde{\pi}|\psi_k)=\frac{1}{\tilde{\pi}(\tau)}H(\tilde{\pi}|\psi_k)$ is convex, so:

$$\pi_k^{\rho,L,M}(1/\tau)H(\tilde{\pi}_k^{\rho,L,M}|\psi_k)\le\alpha\frac{1}{\tilde{\pi}_{0,k}^{\rho,M}(\tau)}H(\tilde{\pi}_{0,k}^{\rho,M}|\psi_k)+(1-\alpha)\frac{1}{\tilde{\eta}_k^{L,M}(\tau)}H(\tilde{\eta}_k^{L,M}|\psi_k).$$

First we consider the second term: since $\psi^n(\{\tau=M\})=\psi^n(\{\tau=L\})=0$, we have:

$$\lim_k\frac{1}{\tilde{\eta}_k^{L,M}(\tau)}H(\tilde{\eta}_k^{L,M}|\psi_k)=\frac{1}{\tilde{\eta}^{L,M}(\tau)}H(\tilde{\eta}^{L,M}|\psi^n)$$

$$\tilde{\eta}^{L,M}(\tau)=\eta^{L,M}(1/\tau)^{-1}$$

$$\tilde{\eta}^{L,M}(d\tau,dW)=\frac{1}{\eta^{L,M}(1/\tau)}\sum_{j=1}^n\frac{1}{\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\mathbb{1}_{\tau\in[M,L)}q_j\psi^n(d\tau|W=j)\otimes\delta_j(dW).$$

We have

$$\begin{aligned}&H(\tilde{\eta}^{L,M}|\psi^n)\\ &=\int_{[M,L)\times\{1,\dots,n\}}\ln\left(\frac{\tilde{\eta}^{L,M}(d\tau,dW)}{\psi^n(d\tau,dW)}\right)\tilde{\eta}^{L,M}(d\tau,dW)\\ &=\frac{1}{\eta^{L,M}(1/\tau)}\sum_{j=1}^n\int_{[M,L)\times\{j\}}\frac{\ln\left(\frac{\tilde{\eta}^{L,M}(d\tau,dW)}{\psi^n(d\tau,dW)}\right)\mathbb{1}_{\tau\in[M,L)}q_j}{\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\psi^n(d\tau|W=j)\otimes\delta_j(dW).\end{aligned}$$

We can decompose ψ^n in the following form:

$$\psi^n(d\tau, dW) = \sum_{j=1}^{n} p_j \psi^n(d\tau|W=j) \otimes \delta_j(d\tau),$$

where $p_j = \psi^n(W=j)$. By definition of Δ^n and $\{1,...,n\}$, we know that $p_i = 0$ implies $q_i = 0$. Then:

$$\begin{aligned}
&H(\tilde{\eta}^{L,M}|\psi^n)\\
=&\frac{1}{\eta^{L,M}(1/\tau)}\times\\
&\sum_{j=1}^{n}\int_{[M,L)\times\{j\}} \ln\left(\frac{1}{\eta^{L,M}(1/\tau)}\frac{\mathbb{1}_{\tau\in[M,L)}q_j}{\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\frac{\psi^n(d\tau|W=j)\otimes\delta_j(dW)}{p_j\psi^n(d\tau|W=j)\otimes\delta_j(d\tau)}\right)\\
&\quad\times\frac{\mathbb{1}_{\tau\in[M,L)}q_j}{\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\psi^n(d\tau|W=j)\otimes\delta_j(dW)\\
=&\frac{1}{\eta^{L,M}(1/\tau)}\sum_{j=1}^{n}q_j\int_{[M,L)}\ln\left(\frac{q_j}{p_j\eta^{L,M}(1/\tau)\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\right)\frac{\psi^n(d\tau|W=j)}{\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\\
=&\frac{1}{\eta^{L,M}(1/\tau)}\sum_{j=1}^{n}q_j\ln\left(\frac{q_j}{p_j\eta^{L,M}(1/\tau)\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\right)\frac{\psi^n(\mathbb{1}_{\tau\in[M,L)}|W=j)}{\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\\
=&\frac{1}{\eta^{L,M}(1/\tau)}\sum_{j=1}^{n}q_j\ln\left(\frac{q_j}{p_j}\right)\frac{\psi^n(\mathbb{1}_{\tau\in[M,L)}|W=j)}{\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\\
&-\frac{1}{\eta^{L,M}(1/\tau)}\sum_{j=1}^{n}q_j\ln\left(\eta^{L,M}(1/\tau)\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)\right)\frac{\psi^n(\mathbb{1}_{\tau\in[M,L)}|W=j)}{\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}.
\end{aligned}$$

Then we study

$$\begin{aligned}
&\frac{1}{\tilde{\eta}_k^{L,M}(\tau)}H(\tilde{\eta}_k^{L,M}|\psi_k)\\
&\quad=\sum_{j=1}^{n}q_j\ln\left(\frac{q_j}{p_j}\right)\frac{\psi^n(\mathbb{1}_{\tau\in[M,L)}|W=j)}{\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\\
&\qquad-\sum_{j=1}^{n}q_j\ln\left(\eta^{L,M}(1/\tau)\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)\right)\frac{\psi^n(\mathbb{1}_{\tau\in[M,L)}|W=j)}{\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}.
\end{aligned}$$

First, we have $\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j) \geq M$, and $\psi^n(\mathbb{1}_{\tau\in[M,L)}|W=j) \leq 1$, so

$$\begin{aligned}
\frac{\psi^n(\mathbb{1}_{\tau\in[M,L)}|W=j)}{\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\left|\sum_{j=1}^{n}q_j\ln\left(\frac{q_j}{p_j}\right)\right| &\leq \frac{1}{M}\left|\sum_{j=1}^{n}q_j\ln\left(\frac{q_j}{p_j}\right)\right|\\
&\underset{M\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Secondly, we have:

$$\sum_{j=1}^{n} q_j \ln\left(\psi^n(\eta^{L,M}(1/\tau)\tau\mathbb{1}_{\tau\in[M,L)}|W=j)\right)\frac{\psi^n(\mathbb{1}_{\tau\in[M,L)}|W=j)}{\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}$$
$$=\sum_{j=1}^{n} q_j \ln\left(\eta^{L,M}(1/\tau)\right)\frac{\psi^n(\mathbb{1}_{\tau\in[M,L)}|W=j)}{\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}$$
$$+\sum_{j=1}^{n} q_j \ln\left(\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)\right)\frac{\psi^n(\mathbb{1}_{\tau\in[M,L)}|W=j)}{\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}$$
$$=\ln\left(\eta^{L,M}(1/\tau)\right)\times\eta^{L,M}(1/\tau)+\sum_{j=1}^{n} q_j\frac{\ln\left(\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)\right)}{\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\psi^n(\mathbb{1}_{\tau\in[M,L)}|W=j).$$

Moreover, $\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)\geq M$, so $\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)\underset{M,L\to+\infty}{\longrightarrow}+\infty$. Since $\dfrac{\ln x}{x}\underset{x\to+\infty}{\longrightarrow}0$ and $\psi^n(\mathbb{1}_{\tau\in[M,+\infty)}|W=j)\underset{M\to+\infty}{\longrightarrow}0$, we have:

$$\sum_{j=1}^{n} q_j\frac{\ln\left(\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)\right)}{\psi^n(\tau\mathbb{1}_{\tau\in[M,L)}|W=j)}\psi^n(\mathbb{1}_{\tau\in[M,L)}|W=j)\underset{M,L\to\infty}{\longrightarrow}0.$$

Eventually, $\eta^{L,M}(1/\tau)=\int_{[M,L)}\frac{1}{\tau}\eta^{L,M}(d\tau)\leq\frac{1}{M}$. Then in particular

$$\lim_{M\to+\infty}\lim_{L\to\infty}\eta^{L,M}\left(\frac{1}{\tau}\right)=0. \tag{2.5.12}$$

Since $x\ln x\underset{x\to0}{\longrightarrow}0$, $\eta^{L,M}(1/\tau)\times\ln\left(\eta^{L,M}(1/\tau)\right)\underset{M,L\to+\infty}{\longrightarrow}0$.

Then

$$\lim_{M\to\infty}\lim_{L\to\infty}\frac{H(\tilde\eta^{L,M}|\psi^n)}{\tilde\eta^{L,M}(\tau)}=0\leq\xi.$$

We focus on the first term. We know that $\tilde\pi^{\rho,M}_{0,k}(\tau)=1/\pi^{\rho,M}_{0,k}(1/\tau)$. We have:

$$\begin{aligned}\lim_{M\to\infty}\lim_{L\to\infty}\lim_{\rho\to0}\lim_{k\to\infty}\pi^{\rho,M}_{0,k}(1/\tau)&=\lim_{M\to\infty}\lim_{L\to\infty}\lim_{\rho\to0}\frac{1}{\beta^{\rho,M}}\sum_{i=1}^{N}\sum_{j=1}^{n}\frac{\tilde\pi(\tau\in A_i,W=j)}{\tilde\pi(A)}\\&=\lim_{M\to\infty}\lim_{L\to\infty}\lim_{\rho\to0}\frac{1}{\beta^{\rho,M}}\\&=\lim_{M\to\infty}\lim_{L\to\infty}\frac{1}{\beta^{M}}\\&=\frac{1}{\tilde\pi(\tau)}.\end{aligned}\tag{2.5.13}$$

Then: $\dfrac{1}{\tilde\pi^{\rho,M}_{0,k}(\tau)}\underset{M,\rho,k}{\rightarrow}\dfrac{1}{\tilde\pi(\tau)}$. Now, we study $H(\tilde\pi^{\rho,M}_{0,k}|\psi_k)$:

As $\pi_0^{\rho,M}(1/\tau) = 1/\beta^{\rho,M}$, we have

$$\begin{aligned}
\tilde{\pi}_0^{\rho,M}(d\tau, dW) &= \frac{1}{\pi_0^{\rho,M}(1/\tau)} \frac{1}{\tau} \pi_0^{\rho,M}(d\tau, dW) \\
&= \beta^{\rho,M} \times \frac{1}{\beta^{\rho,M}} \sum_{i=1}^{N} \sum_{j=1}^{n} \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \psi^n(d\tau, dW | \tau \in A_i, W = j) \\
&= \sum_{i=1}^{N} \sum_{j=1}^{n} \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \psi^n(d\tau, dW | \tau \in A_i, W = j).
\end{aligned}$$

$$\begin{aligned}
&\lim_{k \to +\infty} H(\tilde{\pi}_{0,k}^{\rho,M} | \psi_k) = H(\tilde{\pi}_0^{\rho,M} | \psi^n) \\
=& \sum_{i=1}^{N} \sum_{j=1}^{n} \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \times \\
& \int \ln \left(\sum_{l=1}^{N} \sum_{m=1}^{n} \frac{\tilde{\pi}(\tau \in A_l, W = m)}{\tilde{\pi}(A)} \frac{\psi^n(d\tau, dW | \tau \in A_l, W = m)}{\psi^n(d\tau, dW)} \right) \psi^n(d\tau, dW | \tau \in A_i, W = j) \\
=& \sum_{i=1}^{N} \sum_{j=1}^{n} \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \times \\
& \int_{A_i \times \{j\}} \ln \left(\frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \frac{\psi^n(d\tau, dW)}{\psi^n(\tau \in A_i, W = j) \psi^n(d\tau, dW)} \right) \frac{\psi^n(d\tau, dW)}{\psi^n(\tau \in A_i, W = j)} \\
=& \sum_{i=1}^{N} \sum_{j=1}^{n} \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A) \psi^n(\tau \in A_i, W = j)} \times \int_{A_i \times \{j\}} \ln \left(\frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A) \psi^n(\tau \in A_i, W = j)} \right) \psi^n(d\tau, dW) \\
=& \sum_{i=1}^{N} \sum_{j=1}^{n} \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \ln \left(\frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A) \psi^n(\tau \in A_i, W = j)} \right) \\
=& \sum_{i=1}^{N} \sum_{j=1}^{n} \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \ln \left(\frac{\tilde{\pi}(\tau \in A_i, W = j)}{\psi^n(\tau \in A_i, W = j)} \right) - \ln(\tilde{\pi}(A)) \\
=& \frac{1}{\tilde{\pi}(A)} \sum_{i=1}^{N} \sum_{j=1}^{n} \tilde{\pi}(\tau \in A_i, W = j) \ln \left(\frac{\tilde{\pi}(\tau \in A_i, W = j)}{\psi^n(\tau \in A_i, W = j)} \right) - \ln(\tilde{\pi}(A)) \\
=& \frac{1}{\tilde{\pi}(A)} \left[\tilde{\pi}(\tau \in A^c) \ln \left(\frac{\tilde{\pi}(\tau \in A^c)}{\psi^n(\tau \in A^c)} \right) \right. \\
& \left. + \sum_{i=1}^{N} \sum_{j=1}^{n} \tilde{\pi}(\tau \in A_i, W = j) \ln \left(\frac{\tilde{\pi}(\tau \in A_i, W = j)}{\psi^n(\tau \in A_i, W = j)} \right) \right] \\
& - \left(\ln(\tilde{\pi}(A)) + \tilde{\pi}(A^c) \ln \left(\frac{\tilde{\pi}(A^c)}{\psi^n(A^c)} \right) \right)
\end{aligned}$$

The well known decomposition of entropy property tells us that if $(B_m)_{m \leq M}$ is a partition of $(0,+\infty)\times\{1,...,n\}$ (explicitly proved in 2.B.1):

$$H(\tilde{\pi}|\psi^n) \geq \sum_{m=1}^{M} \tilde{\pi}(B_m)\ln\left(\frac{\tilde{\pi}(B_m)}{\psi^n(B_m)}\right).$$

Here, $(A_i\times\{j\})_{i,j}$ and A^c are a partition of $(0,+\infty)\times\{1,...,n\}$. We have:

$$\begin{aligned}\lim_{k\to+\infty} H(\tilde{\pi}_{0,k}^{\rho,M}|\psi_k) &= H(\tilde{\pi}_0^{\rho,M}|\psi^n)\\ &\leq \frac{1}{\tilde{\pi}(A)}H(\tilde{\pi}|\psi^n) - \left(\ln(\tilde{\pi}(A)) + \tilde{\pi}(A^c)\ln\left(\frac{\tilde{\pi}(A^c)}{\psi^n(A^c)}\right)\right).\end{aligned}$$

Since $\tilde{\pi}(A)\to 1$ and $\tilde{\pi}(A^c)\to 0$ as $M\to\infty$:

$$\limsup_{M\to\infty}\sup_{\rho<1}\lim_{k} H(\tilde{\pi}_{0,k}^{\rho,M}|\psi_k) \leq H(\tilde{\pi}|\psi^n).$$

Now, we can construct a sequence $\pi_k(=\pi_k^{\rho_k,L_k,M_k})$ such that: $\pi_k \rightharpoonup \alpha\pi + (1-\alpha)\delta_\infty\otimes\eta$ and : $\limsup_{k\to\infty}\pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k)\leq I^n(\mu)$.
Eventually, by (2.5.12) and (2.5.13),

$$\begin{aligned}\mu_k\left(\frac{1}{a+b}\right) &= \pi_k\left(\frac{1}{\tau}\right)\\ &= \alpha\pi_{0,k}^{\rho_k,M_k}\left(\frac{1}{\tau}\right) + (1-\alpha)\eta_k^{L_k,M_k}\left(\frac{1}{\tau}\right)\\ &\underset{k\to\infty}{\longrightarrow} \alpha\tilde{\pi}\left(\frac{1}{\tau}\right) = \mu\left(\frac{1}{a+b}\right).\end{aligned}$$

□

2.6 Some additional technical topological lemmata.

Before proving the upper bound and the lower bound of the LDP for empirical measures, we add two useful lemmata. The first one will be used several times for the upper bound. Once again its proof uses conditioning.

Lemma 2.6.1. Let $\mathcal{D}_M^n = \left\{\mu\in\Delta^n, \mu\left(\frac{1}{a+b}\right)\leq M\right\}$. Then $\mathcal{D}_M^n$ is a compact set.

Proof. Since $\mathcal{D}_M^n\subset\mathcal{X}_M$ we already know that it is relatively compact. But since Δ^n is not clearly closed, we have to show that $\mathcal{D}_M^n$ is closed.

Let $(\mu_k)_k$ be a sequence of measures in $\mathcal{D}_M^n$. We will prove that some subsequence again denoted $(\mu_k)_k$ converges in $\mathcal{D}_M^n$. We denote by $\alpha_k, \mu_{0,k}$, π_k and η_k the corresponding quantities in the definition.

For any k, $\alpha_k\in[0,1]$ and $\eta_k\in\mathcal{M}^1(\{1,...,n\})$ which is compact, so there exists a subsequence $\alpha_{\varphi(k)}$, $\eta_{\varphi(k)}$ and $\alpha\in[0,1]$ and $\eta\in\mathcal{M}^1(\{1,...,n\})$ such that $\alpha_{\varphi(k)}\underset{k\to\infty}{\longrightarrow}\alpha$ and $\eta_{\varphi(k)}\rightharpoonup\eta$.

We remind that, for any $r > 0$, $\left\{\nu \in \mathcal{M}^1((0,+\infty]), \nu\left(\frac{1}{x}\right) \leq r\right\}$ is a compact set.

<u>If $\alpha = 0$:</u> then for $f \in \mathcal{C}_b((0,+\infty]^2 \times \{1,...,n\})$:

$$\begin{aligned}\mu_{\varphi(k)}(f) &= \underbrace{\alpha_{\varphi(k)}}_{\to 0} \underbrace{\mu_{0,\varphi(k)}(f)}_{\in[-\|f\|_\infty,\|f\|_\infty]} + \underbrace{(1-\alpha_{\varphi(k)})}_{\to 1}(\delta_{(+\infty,+\infty)} \otimes \underbrace{\eta_{\varphi(k)}}_{\to \eta})(f) \\ &\underset{k\to\infty}{\longrightarrow} (\delta_{(+\infty,+\infty)} \otimes \eta)(f).\end{aligned}$$

Since $(\delta_{(+\infty,+\infty)} \otimes \eta) \in \mathcal{D}_M^n$, $\mu_{\varphi(k)}$ converges in $\mathcal{D}_M^n$.

<u>If $\alpha > 0$:</u> then for k large enough, we have : $|\alpha_{\varphi(k)} - \alpha| < \alpha/2$, so $\alpha_{\varphi(k)} > \alpha/2$. Then, we have:

$$\pi_{\varphi(k)}\left(\frac{1}{\tau}\right) = \mu_{0,\varphi(k)}\left(\frac{1}{a+b}\right) = \frac{1}{\alpha_{\varphi(k)}}\mu_{\varphi(k)}\left(\frac{1}{a+b}\right) \leq \frac{M}{\alpha_{\varphi(k)}} \leq \frac{2M}{\alpha}.$$

We will prove that for all i, $\pi_{\varphi(k)|W=i}$, which is the conditional law π given $W = i$, belongs to $\left\{\nu \in \mathcal{M}^1((0,+\infty]), \nu\left(\frac{1}{x}\right) \leq r\right\}$ for a well-chosen r.

For any k, $\pi_{\varphi(k)} \in \mathcal{M}^1((0,+\infty] \times \{1,...,n\})$, let define $(q_{j,\varphi(k)})_{j\in\{1,...,n\}}$ the weights such that $q_{j,\varphi(k)} = \mathbb{P}_{\pi_{\varphi(k)}}(W = j)$. Then $\pi_{\varphi(k)}(1/\tau)$ can be written with its marginal law:

$$\pi_{\varphi(k)}\left(\frac{1}{\tau}\right) = \sum_{j\in\{1,...,n\}} q_{j,\varphi(k)}\mathbb{E}_{\pi_{\varphi(k)}}\left[\frac{1}{\tau}\middle|W = j\right].$$

The $((q_{j,\varphi(k)})_{j\in\{1,...,n\}})_k$ are at most n sequences of weights in $[0,1]$. There exists a subsequence $(q_{j,\varphi_2(k)})_{j\in\{1,...,n\}}$ and a $(q_j)_{j\in\{1,...,n\}}$ with $q_j \in [0,1]$ such that $\sum q_j = 1$ and $(q_{j,\varphi_2(k)})_{j\in\{1,...,n\}}$ tends to $(q_j)_{j\in\{1,...,n\}}$. Let J the set of indices j such that $q_j \neq 0$. Let $p_{\min}$ the minimum of the q_j for $j \in J$:

$$p_{\min} = \min\{q_j, j \in J\} \neq 0.$$

Then, for k large enough, we have:

$$\forall j \in J,\ q_{j,\varphi_2(k)} > \frac{p_{\min}}{2}.$$

Then, we have, for each $j \in J$

$$\pi_{\varphi_2(k)}\left(\frac{1}{\tau}\right) = \sum_{i\in\{1,...,n\}} q_{i,\varphi_2(k)}\mathbb{E}_{\pi_{\varphi_2(k)}}\left[\frac{1}{\tau}\middle|W = i\right] \geq \frac{p_{\min}}{2}\mathbb{E}_{\pi_{\varphi_2(k)}}\left[\frac{1}{\tau}\middle|W = j\right]$$

and $\pi_{\varphi_2(k)}\left(\frac{1}{\tau}\right) \leq \frac{2M}{\alpha}$, then:

$$\forall j \in J, \mathbb{E}_{\pi_{\varphi_2(k)}}\left[\frac{1}{\tau}\middle|W = j\right] \leq \frac{4M}{\alpha p_{\min}}.$$

So $\forall j \in J, \pi_{\varphi_2(k)|W=j} \in \left\{\nu \in \mathcal{M}^1((0,+\infty]), \nu\left(\frac{1}{x}\right) \leq 4M/(\alpha p_{\min})\right\}$ which is a compact set. There exists a subsequence of $(\pi_{\varphi_2(k)|W=j})_k$ which converges to a π^j for each $j \in J$. By a diagonal argument, we consider an extraction φ_3 such that:

$$\forall j \in J, \pi_{\varphi_3(k)|W=j} \rightharpoonup \pi^j.$$

We define then π corresponding to $(q_j)_j$ and $(\pi^j)_j$:

$$\pi(d\tau, dW) = \sum_{j\in J} q_j \delta_{(W=j)} \pi^j(d\tau)$$
$$\forall f \in \mathcal{C}_b((0,+\infty] \times \{1,...,n\}), \pi(f) = \sum_{j\in J} q_j \pi^j(f(.,j)).$$

Then, $\pi_{\varphi_3(k)} \rightharpoonup \pi$.

It remains to show that μ to belong to $\mathcal{D}^n_M$ i.e to Δ^n.
π can be written as:

$$\pi(d\tau, dW) = \pi(\tau < +\infty) \times \pi(d\tau, dW|\tau < \infty) + (1 - \pi(\tau < \infty))\pi(d\tau, dW|\tau = \infty).$$

Let $\pi^*(d\tau, dW) = \pi(d\tau, dW|\tau < \infty)$, $\beta^* = \pi(\tau < +\infty)$ and η^* such that $\pi(d\tau, dW|\tau = \infty) = \delta_{(+\infty)} \otimes \eta^*$.
Moreover, $\mu_{0,\varphi(k)}$ is the product measure of du and $\pi_{\varphi(k)}$. $\pi_{\varphi_3(k)}$ weakly converges to π, so $\mu_{0,\varphi_3(k)}$ weakly converges to the product measure of du and π. We denote by μ_0 the product measure of du and π^*. Then $\mu_{\varphi(k)}$ tends to

$$\mu = \alpha\beta^*\mu_0 + \alpha(1-\beta^*)\delta_{(+\infty,+\infty)} \otimes \eta^* + (1-\alpha)\delta_{(+\infty,+\infty)} \otimes \eta.$$

With

$$\eta_0 = \frac{\alpha(1-\beta^*)}{(1-\alpha\beta^*)}\eta^* + \frac{(1-\alpha)}{(1-\alpha\beta^*)}\eta,$$

we can write $\mu = \alpha\beta^*\mu_0 + (1-\alpha\beta^*)\delta_{(+\infty,+\infty)} \otimes \eta_0$, so $\mu \in \Delta^n$.

□

The next lemma introduces $\bar{\Delta}^n$, a closed set which contains Δ^n (but is not its closure despite the notation). This will be used directly in the proof of the upper bound of the large deviation principle. A similar set is discussed in [52] lemma 2.5. However we have here to carefully manage the third coordinate, so that the proof in [52] has to be rewritten.

Lemma 2.6.2. The set

$$\begin{aligned}\bar{\Delta}^n := \{\mu \in \mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\}) : \mu = \alpha\mu_0 + (1-\alpha)\delta_{(+\infty,+\infty)} \otimes \eta, \alpha \in [0,1],\\ \mu_0(da, db, dc) = \int_{[0,1]\times(0,\infty)\times\{1,...,n\}} \delta_{(u\tau,(1-u)\tau,W)}(da, db, dc)du \otimes \pi(d\tau, dW),\\ \pi \in \mathcal{M}^1((0,+\infty) \times \{1,...,n\}), \eta \in \mathcal{M}^1(\{1,...,n\})\}\end{aligned} \tag{2.6.1}$$

is closed in $\mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$.

Proof. Let $\mu_k \in \bar{\Delta}^n$ such that μ_k converges to a $\mu \in \mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$ when k tends to ∞. We will prove that μ is in $\bar{\Delta}^n$. Let α_k, π_k and η_k be the corresponding quantities to μ_k as in the definition of $\bar{\Delta}^n$.

$(\alpha_k)_k \in [0,1]$ admits a subsequence which converges to a $\alpha \in [0,1]$. η_k is a sequence of measure on the finite set $\{1,...,n\}$, so a subsequence of η_k converges to $\eta \in \mathcal{M}^1$. As usual we identify the subsequence and the sequence.
$(\pi_k)_k \in \mathcal{M}^1((0,+\infty) \times \{1,...,n\}) \subset \mathcal{M}^1([0,+\infty] \times \{1,...,n\})$ also admits a subsequence which tends to a limit $\pi \in \mathcal{M}^1([0,+\infty] \times \{1,...,n\})$.

Then

$$\begin{aligned}\mu(da,db,dc) = \alpha \int_{[0,1]\times(0,+\infty)\times\{1,...,n\}} &\delta_{(u\tau,(1-u)\tau,W)}(da,db,dc)du \otimes \pi(d\tau,dW) \\ &+ (1-\alpha)\delta_{(+\infty,+\infty)}(da,db) \otimes \eta(dc).\end{aligned}$$

We need to prove that μ is in $\bar{\Delta}^n$. First we verify that π has no weight in 0, and secondly we prove that by rewritting μ, we can consider $\bar{\pi} \in \mathcal{M}^1((0,+\infty) \times \{1,...,n\})$, $\bar{\alpha}$ and $\bar{\eta}$ such that μ have the good form.

First, we consider the weight of 0 for π. By Skorohod's representation theorem, there exists a sequence $(P_k,V_k)_k$ and X_k of random variables such that $(P_k,V_k) \in (0,+\infty) \times \{1,...,n\}$ has the law π_k, X_k has the law η_k and (P_k,V_k) converges a.s. to (P,V) of law π, X_k converges a.s. to X of law η. Let U be an uniform random variable independent of $(P_k)_k$ and P. For any $f \in \mathcal{C}_b([0,+\infty]^2 \times \{1,...,n\})$ we obtain

$$\begin{aligned}\mu_k(f) &= \alpha_k \mathbb{E}[f(UP_k,(1-U)P_k,V_k)] + (1-\alpha_k)\mathbb{E}[f(+\infty,+\infty,X_k)] \\ &\underset{k\to\infty}{\longrightarrow} \alpha\mathbb{E}[f(UP,(1-U)P,V)] + (1-\alpha)\mathbb{E}[f(+\infty,+\infty,X)]\end{aligned}$$

and this limit is equal to $\mu(f)$. For $f_\varepsilon(a,b,c) = \mathbb{1}_{a+b<\varepsilon}$, since $\mu \in \mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$: $\alpha\pi(\{P<\varepsilon\}) = \mu(f_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0$. Then $\pi \in \mathcal{M}^1((0,+\infty] \times \{1,...,n\})$.

Now we prove that we can write μ in the same form that in $\bar{\Delta}^n$. Let $\beta = \pi(P=+\infty)$. If $\beta = 0$, then

$$\alpha \int_{[0,1]\times(0,+\infty)\times\{1,...,n\}} \delta_{(u\tau,(1-u)\tau,W)}(da,db,dc)du \otimes \pi(d\tau,dW) + (1-\alpha)\delta_{(+\infty,+\infty)}(da,db) \otimes \eta(dc)$$

is already in the good form.
Else,

$$\begin{aligned}\mu(f) &= \alpha\beta\mathbb{E}[f(UP,(1-U)P,V)|P<\infty] + \alpha(1-\beta)\mathbb{E}[f(+\infty,+\infty,V)|P=\infty] \\ &\quad + (1-\alpha)\mathbb{E}[f(+\infty,+\infty,X)] \\ &= \alpha\beta\mathbb{E}[f(UP,(1-U)P,V)|P<\infty] + \\ &\quad (1-\alpha\beta)\left(\frac{\alpha(1-\beta)}{1-\alpha\beta}\mathbb{E}[f(+\infty,+\infty,V)|P=\infty] + \frac{1-\alpha}{1-\alpha\beta}\mathbb{E}[f(+\infty,+\infty,X)]\right).\end{aligned}$$

Let $\bar{\eta}(dW) = \frac{\alpha(1-\beta)}{1-\alpha\beta}\pi(dW|\tau=\infty) + \frac{1-\alpha}{1-\alpha\beta}\eta$, $\bar{\alpha} = \alpha\beta$ and $\bar{\pi}(d\tau,dW) = \pi(d\tau,dW|\tau<\infty)$. Therefore $\mu \in \bar{\Delta}^n$. □

Remark 2.6.3. Here again one can replace $\{1, ..., n\}$ by a compact subset $\mathbb{W}$. $\diamondsuit$

We finally set two important results. The first one is identical to Lemma 2.6 and Lemma 2.7 in [52]:

Lemma 2.6.4. If $\mu \in \Delta^n$ then $I^n(\mu) \leq \sup_{f\in\Gamma} \mu(f)$ and if $\mu \in \bar{\Delta}^n \backslash \Delta^n$ then $\sup_{f\in\Gamma} \mu(f) = +\infty$. (recall that Γ is defined in (2.4.9))

The proof of the Lemma is almost identical to [52] (see Lemma 2.A.5 and Lemma 2.A.6). Indeed it is enough to consider a third variable $c \in \{1, ..., n\}$. Similarly to the proof of Lemma 2.6 we have to introduce a function

$$f_{d,\varphi,M}(a, b, c) = \frac{\varphi(a + b, c)}{a + b} + d\mathbb{1}_{(M,+\infty]}(a + b)$$

for a continuous and compactly supported φ. $f_{d,\varphi,M}$ is lower semi-continuous. One can then copy the proof of Lemma 2.6 in [52]. For the second part we may consider the same function f_ε as in the proof of Lemma 2.7 in [52].

Notice that again we may replace $\{1, ..., n\}$ by a compact $\mathbb{W}$.

The second one (and its proof) is identical to Proposition 3.3 in [52] (see Proposition 2.A.7)

Lemma 2.6.5. For all $f \in \Gamma$ and all $t > 0$, $\mathbb{E}(e^{t\,\mu_t^n(f)}) \leq \frac{D_{n,f}}{1-C_{n,f}} < +\infty$.

2.7 Proof of Theorem 2.4.3.

In this section, we prove the LDP for $\mathbb{P}_t^n$ that denotes the $\mathbb{P}$ distribution of μ_t^n. This time the introduction of the third coordinate W replacing $F(\tau)$ does not create any new difficulty. However some points in the proofs of [52] are not clear for us and we will give the details for some points.

The proof of the upper bound is made in several steps: the proof of a weak principle, for compact sets $\mathcal{C}$ (itself divided in several steps), and the proof that μ_t^n is an exponentially tight family, i.e. satisfies: for all $\alpha \in \mathbb{R}^+$ there exists some compact set $\mathcal{K}_\alpha$ with

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}_t^n(\mathcal{K}_\alpha^c) < -\alpha\,.$$

The (full) upper bound for closed sets $\mathcal{C}$ then follows from [52]. These steps are described in [52] section 3, and are given in Appendix 2.A.3.1.

Exponential tightness is the aim of Lemma 3.1 in [52]. The first step is the following lemma

Lemma 2.7.1. We have:

$$\lim_{M\to+\infty} \limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\mu_t^n\left(\frac{1}{a+b}\right) > M\right) = -\infty.$$

whose proof is unchanged in our case (see Lemma 2.A.8). Since $\{\nu \in \mathcal{M}_1((0,+\infty]^2 \times \{1,...,n\}), \nu(1/(a+b)) \leq M\}$ is compact (see lemma 2.4.1), exponential tightness follows.

The proof of the upper bound for compact subsets is done in [52] p. 2261 and 2262 beginning with the definition of the set

$$\Delta_{M,g,\delta} = \{\mu \in \mathcal{M}_1((0,+\infty]^2), \exists \nu \in \Delta, |\nu(g) - \mu(g)| \leq \delta, \mu(1/(a+b)) \leq M\}$$

for some continuous and bounded g, δ and M positive. We do not see an immediate argument showing that this set is closed (hence compact since it is relatively compact). We will thus slightly modify the proof in [52].

We introduce a modified set for M δ and g as before,

$$\begin{aligned} \Delta^n_{M,g,\delta} = \quad & \{\mu \in \mathcal{M}_1((0,+\infty]^2 \times \{1,...,n\}), \exists \nu \in \Delta^n, |\nu(g) - \mu(g)| \leq \delta, \\ & \mu(1/(a+b)) \leq M, \nu(1/(a+b)) \leq M+\delta\}. \end{aligned} \tag{2.7.1}$$

We also define

$$R^n_{M,g,\delta} := -\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}^n_t((\Delta^n_{M,g,\delta})^c). \tag{2.7.2}$$

One can of course replace g by a finite number of continuous and bounded g_i's.

Lemma 2.7.2. $\Delta^n_{M,g,\delta}$ is a compact set.

Proof. Notice that if $\mu_k \in \Delta^n_{M,g,\delta}$ weakly converges to some μ, $\mu \in \mathcal{D}^n_M$ according to lemma 2.6.1. The corresponding sequence ν_k in Δ^n actually belongs to $\mathcal{D}^n_{M+\delta}$ so that one can find a subsequence still denoted ν_k that converges to some $\nu \in \mathcal{D}^n_{M+\delta}$ (this is the key difference with $\Delta_{M,g,\delta}$). Since g is bounded and continuous, by taking limits we have $|\nu(g) - \mu(g)| \leq \delta$. Of course for $\varepsilon > 0$,

$$\nu(1/(a+b+\varepsilon)) = \lim_k \nu_k(1/(a+b+\varepsilon)) \leq \lim_k \nu_k(1/(a+b)) \leq M+\delta,$$

and $\nu(1/(a+b)) \leq M+\delta$ follows by letting ε go to 0 thanks to the monotone convergence theorem. Compactness follows since $\Delta^n_{M,g,\delta} \subset \mathcal{D}^n_M$ which is compact. $\square$

The next step consists in showing that the empirical measure μ^n_t is close to Δ^n more precisely

Lemma 2.7.3. For $f \in C_b((0,+\infty]^2 \times \{1,...,n\})$ we define

$$\nu^n_t(f) := \frac{1}{t}\sum_{i=1}^{M_t} \tau_i \int_0^1 f(u\tau_i, (1-u)\tau_i, W^n_i)du + \frac{t - S_{M_t}}{t} f(+\infty, +\infty, W^n_{M_t+1}).$$

Then $\nu^n_t \in \Delta^n$ almost surely.

For all g, δ there exists some $t(g,\delta)$ such that for $t \geq t(g,\delta)$, the events $\{|\mu^n_t(g) - \nu^n_t(g)| > \delta\}$ and $\{\left|\mu^n_t\left(\frac{1}{a+b}\right) - \nu^n_t\left(\frac{1}{a+b}\right)\right| > \delta\}$ are almost surely empty.

Proof. This lemma is the analogue of Lemma 3.2 in [52] but due to the modification of our $\Delta^n_{M,g,\delta}$ we have to complete the proof therein.

Arguing as in the proof of Lemma 3.2 in [52] one shows that for t large enough the set $\{|\mu^n_t(g) - \nu^n_t(g)| > \delta\}$ is almost surely empty (see Lemma 2.A.9). For the other term,

$$\left|\mu^n_t\left(\frac{1}{a+b}\right) - \nu^n_t\left(\frac{1}{a+b}\right)\right|$$

$$= \left|\left(\frac{1}{t}\sum_{i=1}^{M_t}\tau_i \times \frac{1}{\tau_i} + \frac{\tau_{M_t+1}}{t} \times \frac{t-S_{M_t}}{\tau_{M_t+1}} \times \frac{1}{\tau_{M_t+1}}\right) - \frac{1}{t}\sum_{i=1}^{M_t}\tau_i \times \frac{1}{\tau_i}\right|$$
$$= \frac{1}{t}\,\frac{t-S_{M_t}}{\tau_{M_t+1}} \le \frac{1}{t}\,.$$

Hence for $t > 1/\delta$ we have $\left|\mu^n_t\left(\frac{1}{a+b}\right) - \nu^n_t\left(\frac{1}{a+b}\right)\right| \le \delta$. □

Corollary 2.7.4. $\lim_{M\to+\infty} R^n_{M,\delta,g} = +\infty$.

Proof. According to lemma 2.7.3, for t large enough (that does not depend on M but only on δ and g) the set $\{\mu^n_t(1/(a+b)) \le M\} \cap (\Delta^n_{M,g,\delta})^c$ is almost surely empty. Hence for t large enough

$$\mathbb{P}^n((\Delta^n_{M,g,\delta})^c) \le \mathbb{P}^n(\mu^n_t(1/(a+b)) > M)$$

and the result follows from lemma 2.7.1.

□

We may now follow the proof in [52]. First exactly as in [52], thanks to Lemma 2.6.5, for all open set $\mathcal{O}$, all g, M, δ and all $f \in \Gamma$,

$$\limsup_{t\to+\infty}\frac{1}{t}\ln \mathbb{P}^n_t(\mathcal{O}) \le -\inf_{\nu\in\mathcal{O}} I^n_{f,M,g,\delta}(\mu)$$

where

$$I^n_{f,M,g,\delta}(\mu) = \begin{cases}\mu(f) \wedge R^n_{M,g,\delta} & \text{if } \mu \in \Delta^n_{M,g,\delta}\\ +\infty & \text{otherwise.}\end{cases}, \tag{2.7.3}$$

Since f is lower semicontinuous, $I^n_{f,M,g,\delta}(\mu)$ is also lower semicontinuous thanks to the compactness of $\Delta^n_{M,g,\delta}$. One can thus deduce as in [52] that for all compact subset $\mathcal{K}$,

$$\limsup_{t\to+\infty}\frac{1}{t}\ln \mathbb{P}^n_t(K) \le -\inf_{\mu\in\mathcal{K}}\sup_{f,M,g,\delta} I^n_{f,M,g,\delta}(\mu) := -\inf_{\mu\in\mathcal{K}}\tilde{I}(\mu)\,.$$

If $\mu \notin \bar{\Delta}^n$ which is closed according to lemma 2.6.2, one can find an open neighborhood $\mathcal{U}$ of μ such that $\mathcal{U} \cap \bar{\Delta}^n = \emptyset$. We may choose $\mathcal{U}$ of the form

$$\mathcal{U} = \bigcap_{i=1}^{k}\{\nu, |\nu(g_i) - \mu(g_i)| \le \delta\}$$

for some family $g_1, ..., g_k$ of bounded and continuous functions, so that for at least one of the g_i's denoted by g, $\mu \notin \Delta^n_{M,g,\delta}$. Hence $\cap_{g,\delta} \Delta^n_{M,g,\delta} \subset \bar{\Delta}^n$ and $\tilde{I}(\mu) = +\infty$ if $\mu \notin \bar{\Delta}^n$. Together with corollary 2.7.4 we deduce that

$$\tilde{I}(\mu) \geq \sup_{f \in \Gamma} I_f(\mu) \quad \text{where} \quad I_f(\mu) = \mu(f) \text{ if } \mu \in \bar{\Delta}^n \ , \ I_f(\mu) = +\infty \text{ otherwise.}$$

Thus, according to lemma 2.6.4, $\tilde{I}(\mu) \geq I^n(\mu)$ and the upper bound is proved.

The proof of the lower bound is similar to [52], just replacing the sample τ_i by a sample (τ_i, W_i), and thus is omitted. (See Appendix 2.A.3.2 for the totality of the proof.)

2.8 Proof of Theorem 2.4.4.

We turn to the proof of theorem 2.4.4. We first will deduce a LDP from theorem 2.4.3 by using the contraction principle, and then identify the rate function.

2.8.1 The contraction principle.

Define $\varphi(a, b, c) = \frac{c}{a+b}$ which is continuous on $(0, +\infty]^2 \times \{1, ..., n\}$. Then

$$\mu^n_t(\varphi) = \frac{1}{t} \sum_{i=1}^{M_t} W^n_i \ + \ \frac{(t - S_{M_t}) W^n_{M_t+1}}{t \, \tau_{M_t+1}} \, .$$

Remark that

$$\left| \frac{(t - S_{M_t}) W^n_{M_t+1}}{t \, \tau_{M_t+1}} \right| \leq \frac{n}{t}$$

so that

$$\limsup_{t \to \infty} \frac{1}{t} \ln \mathbb{P} \left(\left| \mu^n_t(\varphi) - \frac{1}{t} \sum_{i=1}^{M_t} W^n_i \right| > \delta \right) = -\infty$$

showing that $\mu^n_t(\varphi)$ and $\frac{1}{t} \sum_{i=1}^{M_t} W^n_i = Z^n_t / t$ will satisfy the same LDP. The contraction principle (Theorem 4.2.1 in [52]) should thus furnish some LDP for Z^n_t/t. Unfortunately φ is not bounded so that $\mu \mapsto \mu(\varphi)$ is not continuous from $\mathcal{M}_1(0, +\infty]^2 \times \{1, ..., n\})$ to $\mathbb{R}$ and the contraction principle does not apply directly. We are thus obliged one more time to use an approximation procedure replacing φ by

$$\varphi^\varepsilon(a, b, c) = \frac{c}{(a + b) \vee \varepsilon}$$

for $\varepsilon > 0$. Replacing τ_i by $\tau^\varepsilon_i = \tau_i \vee \varepsilon$ we may introduce M^ε_t, A^ε_t, B^ε_t, $C^{n,\varepsilon}_t$ and $\mu^{n,\varepsilon}_t$ as in subsection 2.4.1. The results of the previous section apply to this new process (introducing a rate function $I^{n,\varepsilon}(\mu)$) This time we may apply the contraction principle so that $\mu^{\varepsilon,n}_t(\varphi^\varepsilon)$ satisfies a LDP with rate function

$$J^{\varepsilon,n}(m) = \inf\{I^{n,\varepsilon}(\mu) \ , \ \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, ..., n\}), m = \mu(\varphi^\varepsilon)\} \, . \tag{2.8.1}$$

We now have

$$\mu_t^{\varepsilon,n}(\varphi^\varepsilon) = \frac{1}{t}\sum_{i=1}^{M_t^\varepsilon} W_i^n + \frac{(t - S^\varepsilon_{M_t^\varepsilon})W^n_{M_t^\varepsilon+1}}{t\,\tau^\varepsilon_{M_t^\varepsilon+1}},$$

so that, since $M_t^\varepsilon \leq M_t$,

$$\begin{aligned}\left|\mu_t^{\varepsilon,n}(\varphi^\varepsilon) - \frac{1}{t}\sum_{i=1}^{M_t} W_i^n\right| &\leq \frac{1}{t}\left|\sum_{i=M_t^\varepsilon+1}^{M_t} W_i^n\right| + \left|\frac{(t - S^\varepsilon_{M_t^\varepsilon})W^n_{M_t^\varepsilon+1}}{t\,\tau^\varepsilon_{M_t^\varepsilon+1}}\right| \\ &\leq \frac{n}{t}\left((M_t - M_t^\varepsilon) + 1\right).\end{aligned}$$

But according to [52] Lemma 5.4, M_t^ε is an exponentially good approximation of M_t (see Lemma 2.A.13). It follows that $\mu_t^{\varepsilon,n}(\varphi^\varepsilon)$ is an exponentially good approximation of Z_t^n/t (see Lemma 2.A.14) so that, finally, thanks to Theorem 2.2.2, Z_t^n/t satisfies a weak LDP with rate function

$$\tilde{J}^n(m) = \sup_{\delta>0}\liminf_{\varepsilon\to 0}\inf_{|z-m|<\delta} J^{\varepsilon,n}(m). \tag{2.8.2}$$

In particular we know that $\tilde{J}^n$ is l.s.c. so that its level sets are closed.

2.8.2 Study of the rate function.

The goal of this subsection is to show the following lemma (partly close to Lemma 5.1 in [52]) explaining the various forms of the rate function (recall that $\varphi(a,b,c) = c/(a+b)$).

Lemma 2.8.1. We define, for all $m > 0$, $\bar{J}^n(m) := \inf\{I^n(\mu), \mu \in \mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\}), m = \mu(\varphi)\}$. Then

i) $\bar{J}^n = J^n$ (J^n is defined in (2.2.6)). In addition

$$\bar{J}^n(m) = \inf\{I_0^n(\mu), \mu \in \mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\}), m = \mu(\varphi)\}.$$

ii) We also have:

$$\begin{aligned}\bar{J}^n(m) &= \min\{I^n(\mu), \mu \in \mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\}), m = \mu(\varphi)\} &(2.8.3)\\ &= \min\{I_0^n(\mu), \mu \in \mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\}), m = \mu(\varphi)\}. &(2.8.4)\end{aligned}$$

iii) Finally,

$$J^n = \bar{J}^n = \tilde{J}^n.$$

Proof. Proof of i) We have

$$\begin{aligned}
\bar{J}^n(m) &:= \inf\left\{I^n(\mu), \mu \in \mathcal{M}_1((0,+\infty]^2 \times \{1,...,n\}), \int \frac{c}{a+b}\mu(da,db,dc) = m\right\}\\
&= \inf\left\{I^n(\mu), \mu \in \Delta^n, \int \frac{c}{a+b}\mu(da,db,dc) = m\right\}\\
&= \inf\left\{\alpha\pi\left(\frac{1}{\tau}\right)H(\tilde{\pi}|\psi^n) + (1-\alpha)\xi, \alpha \in [0,1], \mu_0 \in \Delta_0^n,\right.\\
&\qquad \left.\alpha\int \frac{c}{a+b}\mu_0(da,db,dc) = m\right\}\\
&= \inf\left\{\alpha\pi\left(\frac{1}{\tau}\right)H(\tilde{\pi}|\psi^n) + (1-\alpha)\xi, \alpha \in [0,1], \mu_0 \in \Delta_0^n, \alpha\pi\left(\frac{W}{\tau}\right) = m\right\}\\
&= \inf\left\{\alpha\frac{1}{\tilde{\pi}(\tau)}H(\tilde{\pi}|\psi^n) + (1-\alpha)\xi, \alpha \in [0,1], \tilde{\pi} \in \mathcal{M}^1((0,+\infty) \times \{1,...,n\}),\right.\\
&\qquad \left.\alpha\frac{\tilde{\pi}(W)}{\tilde{\pi}(\tau)} = m\right\}\\
&= \inf\left\{\frac{\alpha}{\beta}H(\tilde{\pi}|\psi^n) + (1-\alpha)\xi, \alpha \in [0,1], \beta > 0, \tilde{\pi}(\tau) = \beta, \alpha\frac{\tilde{\pi}(W)}{\beta} = m\right\}
\end{aligned}$$

Let $p(a,b) = \inf\{H(\nu|\psi^n), \nu(\tau) = a, \nu(W) = b\}$. We have $p = \Lambda_n^*$ according to Csiszar I-projection theorem (Theorem 3 in [23]). As in [52] (proof of Lemma 5.1) another way is to directly prove the dual equality $p^* = \Lambda^n$.

We thus have

$$\bar{J}^n(m) = \inf\left\{\frac{\alpha}{\beta}\Lambda_n^*\left(\beta, \frac{m\beta}{\alpha}\right) + (1-\alpha)\xi, \alpha \in [0,1], \beta > 0\right\}.$$

But

$$\begin{aligned}
\frac{\alpha}{\beta}\Lambda_n^*\left(\beta, \frac{m\beta}{\alpha}\right) &= \sup_{x,y}\left\{\alpha x + my - \frac{\alpha}{\beta}\Lambda_n(x,y)\right\}\\
&= \beta'\Lambda_n^*\left(\frac{\alpha}{\beta'}, \frac{m}{\beta'}\right) \text{ where } \beta' = \frac{\alpha}{\beta}.
\end{aligned}$$

Thus

$$\bar{J}^n(m) = \inf\left\{\beta\Lambda_n^*\left(\frac{\alpha}{\beta}, \frac{m}{\beta}\right) + (1-\alpha)\xi, \alpha \in [0,1], \beta > 0\right\}.$$

We will show that :

$$\inf_{\alpha\in[0,1]}\left\{\beta\Lambda_n^*\left(\frac{\alpha}{\beta}, \frac{m}{\beta}\right) + (1-\alpha)\xi\right\} = \beta\Lambda_n^*\left(\frac{1}{\beta}, \frac{m}{\beta}\right).$$

Taking $\alpha = 1$, we see that the left hand side is less than or equal to the right hand side. To show the converse inequality, pick $\alpha \in [0,1]$:

$$\begin{aligned}
\beta\Lambda_n^*\left(\frac{\alpha}{\beta}, \frac{m}{\beta}\right) + (1-\alpha)\xi &= \sup_{x,y}\{\alpha x + (1-\alpha)\xi + my - \beta\Lambda_n(x,y)\}\\
&\geq \sup_{x,y}\{x \wedge \xi + my - \beta\Lambda_n(x,y)\}.
\end{aligned}$$

Since W is bounded, $e^{yW} \geq C(y) > 0$ for all y, so that we have for all $x > \xi$ and all y,

$$\psi^n(\mathrm{e}^{x\tau + yW}) \geq C(y)\,\psi^n(\mathrm{e}^{x\tau}) = +\infty\,.$$

This shows that $\Lambda_n(x,y) = +\infty$, for all $x > \xi$ and for all y. Hence, the supremum on x can be restricted to the supremum on $\{x \leq \xi\}$:

$$\begin{aligned}\beta\Lambda_n^*\left(\frac{\alpha}{\beta},\frac{m}{\beta}\right) + (1-\alpha)\xi &\geq \sup_{x,y}\{x \wedge \xi + my - \beta\Lambda_n(x,y)\}\\ &= \sup_{x\leq\xi,y}\{x + my - \beta\Lambda_n(x,y)\}\\ &= \beta\Lambda_n^*\left(\frac{1}{\beta},\frac{m}{\beta}\right)\end{aligned}$$

and the desired inequality is proved.

Notice that during the proof we have seen than the minimization is obtained looking only at $\mu \in \Delta_0^n$ so that we may replace I^n by I_0^n in the definition of $\bar{J}^n$.

Proof of ii). This part is completely similar to the corresponding one in the proof of Lemma 5.1 in [52]. The only thing to see is that we may replace $||F||_\infty$ in [52] by K if $|W| \leq K$.

Proof of iii). To prove this equality, we first prove the inequality $\bar{J}^n(m) \leq \tilde{J}^n(m)$.

Let $m \in \mathbb{R}^+$ and $\varepsilon > 0$. First, remark that $\bar{J}^n(m) \leq J^{n,\varepsilon}(m)$. Indeed, if $\mu \in \Delta^n$, with associated π, α and η, is such that $H(\tilde{\pi}|\psi^{n,\varepsilon}) < +\infty$ and $\mu(\varphi^\varepsilon) = m$, then $\tilde{\pi}$ has its support included in $(\varepsilon,+\infty)\times\{1,...,n\}$. Hence $I^n(\mu) = I^{n,\varepsilon}(\mu)$. Moreover, $\mu(\varphi) = \mu\left(\frac{1}{a+b}\right) = m$. This yields

$$\begin{aligned}\bar{J}^n(m) &= \inf\{I^n(\mu), \mu \in \mathcal{M}^1((0,+\infty]^2\times\{1,...,n\}), m = \mu(\varphi)\}\\ &\leq \inf\{I^n(\mu), \mu \in \mathcal{M}^1(\{(a,b,c) \in (0,+\infty)^2\times\{1,...,n\}), \mu(\{a+b\leq\varepsilon\}) = 0,\\ &\qquad m = \mu(\varphi)\}\\ &= \inf\{I^{n,\varepsilon}(\mu), \mu \in \mathcal{M}^1((0,+\infty]^2\times\{1,...,n\}), m = \mu(\varphi^\varepsilon)\} = J^{n,\varepsilon}.\end{aligned}$$

We will take the limit as $\varepsilon \to 0$. To this end we may write for $\delta > 0$,

$$\inf_{z,|z-m|\leq\delta}\bar{J}^n(z) \leq \inf_{z,|z-m|\leq\delta}J^{n,\varepsilon}(z)$$

so that

$$\inf_{z,|z-m|\leq\delta}\bar{J}^n(z) \leq \liminf_{\varepsilon\to0}\inf_{z,|z-m|\leq\delta}J^{n,\varepsilon}(z) \leq \tilde{J}^n(m)$$

Since $\bar{J}^n = J^n$, it is clearly lower semicontinuous so that

$$\bar{J}^n(m) \leq \liminf_{\delta\to0}\inf_{z,|z-m|\leq\delta}\bar{J}^n(z) \leq \tilde{J}^n(m)\,.$$

Finally $\bar{J}^n \leq \tilde{J}^n$

Now, we prove that $\bar{J}^n \geq \tilde{J}^n$. Since $\bar{J}^n(m)$ is a minimum, let $\bar{\mu}$ a measure such that $\bar{J}^n(m) = I^n(\bar{\mu})$ and $\bar{\mu}(\varphi) = m$. For a sequence ε_k going to 0, we consider ψ_k

the distribution of $(\tau \vee \varepsilon_k, W)$. Using proposition 2.5.6, there exists $\mu_k \in \Delta_0^n$ with associated to ψ_k such that $\mu_k \rightharpoonup \bar{\mu}$ and $\limsup_{k\to\infty} I_k^n(\mu_k) \leq I^n(\bar{\mu})$. Here $I_k^n = I^n$, so $\limsup_{k\to\infty} I^n(\mu_k) \leq I^n(\bar{\mu})$.

By construction $\mu_k(\{a+b<\varepsilon_k\}) = 0$ and for $\tilde{\pi}_k$ associated to μ_k, we have $H(\tilde{\pi}_k|\psi^n) < +\infty$. Since $\tilde{\pi}_k$ is a measure on $(\varepsilon_k, +\infty) \times \{1, ..., n\}$, $H(\tilde{\pi}_k|\psi_k) = H(\tilde{\pi}_k|\psi^n)$, and it follows $I^n(\mu_k) = I^{n,\varepsilon_k}(\mu_k)$.

As a consequence of proposition 2.5.6 it holds $\mu_k\left(\frac{1}{a+b}\right) \to \bar{\mu}\left(\frac{1}{a+b}\right)$. Hence, $\delta_k = \left|\mu_k\left(\frac{1}{a+b}\right) - \bar{\mu}\left(\frac{1}{a+b}\right)\right| \to 0$.

We can now write

$$\inf_{z,|z-m|\leq\delta_k} J^{n,\varepsilon_k}(z) \leq I^{n,\varepsilon_k}(\mu_k) = I^n(\mu_k)$$

so that

$$\limsup_{k\to\infty} \inf_{z,|z-m|\leq\delta_k} J^{n,\varepsilon_k}(z) \leq \limsup_{k\to\infty} I^n(\mu_k) \leq I^n(\bar{\mu}) = \bar{J}^n(m)$$

and finally

$$\tilde{J}^n(m) = \sup_{\delta>0} \liminf_{\varepsilon\to 0} \inf_{z,|z-m|<\delta} J^{n,\varepsilon}(z) \leq \bar{J}^n(m)\,.$$

□

Remark 2.8.2. The first two items i) and ii) in the previous lemma are still hold true if we replace $\{1, ..., n\}$ by a compact set $\mathbb{W}$. ♢

2.8.3 End of the proof.

At this point we have obtained that Z_t^n/t satisfies a weak LDP with rate function J^n. In order to get the full LDP we have to show that the level sets are bounded (since we know that they are closed). Recall that it is not a direct consequence of the contraction principle since $\mu \mapsto \mu(\varphi)$ is not continuous.

First since $J^n \leq J^{n,\varepsilon}$ for all $\varepsilon > 0$, for all closed set F it holds

$$\limsup_{\varepsilon\to 0} \inf_{y\in F} J^{n,\varepsilon}(y) \geq \inf_{y\in F} J^n(y).$$

It remains to show that J^n is a good rate function i.e. that it has compact level sets. According to the previous subsection for a sequence m_k such that $J^n(m_k) \leq \beta$, we may find a sequence $\mu_k \in \Delta_0^n$ such that $\mu_k(\varphi) = m_k$ and $I_0^n(\mu_k) \leq \beta$. The corresponding π_k satisfies $\limsup \pi_k(1/(a+b)) < +\infty$ according to proposition 2.5.3 (here $\alpha_k = 1$). Since $|\varphi(a,b,c)| \leq C/(a+b)$, m_k is bounded and one can find a convergent subsequence.

2.9 Proof of Theorem 2.2.5.

When W is bounded and discrete, the full LDP is already given by Theorem 2.4.4. If W isn't bounded, or isn't discrete, we need to differentiate whether $\beta_0 = \infty$ or not.
In the first case, when $\beta_0 = \infty$, we are able to prove a Large Deviation principle. Since the rate function $\tilde{J}$ is difficult to calculate, we compare $\tilde{J}$ with an other function J to simplify some inequalities. This work is done in the following subsection 2.9.1.
In the second case, since the approximation Z_t^n/t isn't an exponentially good approximation of Z_t/t, we only prove some useful inequalities of deviations, but we're not able to prove the Large Deviation Principle.

2.9.1 Case A: $\beta_0 = \infty$

In this case, Z_t^n/t is an exponentially good approximation of Z_t/t (see Lemma 2.3.1, 2.3.2 or 2.3.4 depending on the approximation strategy). Combining the LDP principle obtained for Z_t^n/t (Theorem 2.4.4) with Theorem 2.2.2 (1) we obtain that Z_t/t satisfies a weak LDP with rate function

$$\tilde{J}(m) = \sup_{\delta>0} \liminf_{n\to\infty} \inf_{|m-z|<\delta} J^n(z)\,.$$

In order to obtain a full LDP, we use lemma 2.2.3. Therefore it remains to show that Z_t/t is exponentially tight.

Lemma 2.9.1. Assume that Assumption 2.2.4 is fulfilled, then $(Z_t/t)_{t\geq 0}$ is exponentially tight, i.e. for all $\alpha > 0$, there exists a compact set K_α such that

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{Z_t}{t} \notin K_\alpha^c\right) < -\alpha.$$

Proof. Since Z_t^n/t is an approximation of Z_t/t and satisfies a full LDP, we can decompose the probability as following: for each n, and for all δ:

$$\begin{aligned}
&\mathbb{P}\left(\frac{Z_t}{t} \notin [-A, A]\right)\\
&\leq \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{Z_t^n}{t}\right| > \delta\right) + \mathbb{P}\left(\frac{Z_t^n}{t} \notin [-A+\delta, A-\delta]\right)\\
&\leq \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{Z_t^n}{t}\right| > \delta\right) + \mathbb{P}\left(\frac{Z_t^n}{t} < -A+\delta\right) + \mathbb{P}\left(\frac{Z_t^n}{t} > A-\delta\right).\\
&\leq 3\max\left(\mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{Z_t^n}{t}\right| > \delta\right), \mathbb{P}\left(\frac{Z_t^n}{t} < -A+\delta\right), \mathbb{P}\left(\frac{Z_t^n}{t} > A-\delta\right)\right).
\end{aligned} \tag{2.9.1}$$

By Lemma 2.3.1, Z_t^n/t and Z_t/t satisfies

$$\forall\delta > 0, \lim_n \overline{\lim_t} \frac{1}{t} \ln \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{Z_t^n}{t}\right| > \delta\right) = -\frac{\beta_0\delta}{2},$$

i.e.

$$\forall \alpha > 0, \forall \delta > \frac{2\alpha}{\beta_0}, \exists n(\alpha,\delta), \forall n > n(\alpha,\delta), \overline{\lim_t} \frac{1}{t} \ln \mathbb{P}\left(\left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| > \delta \right) \leq -\alpha. \tag{2.9.2}$$

(If $\beta_0 = +\infty$, we can consider all $\delta > 0$.)

We just have to study $\mathbb{P}\left(\frac{Z_t^n}{t} > A - \delta\right)$ and the symmetric case. We know from Theorem 2.4.4 that:

$$\overline{\lim_t} \frac{1}{t} \ln \mathbb{P}\left(\frac{Z_t^n}{t} > B \right) \leq - \inf_{m > B} J^n(m).$$

It remains to show that $\forall \alpha > 0$ one can choose a level B_α such that $\forall m > B_\alpha, J^n(m) > \alpha$.

Remind that

$$\begin{aligned} J^n(m) &= \inf_{\beta > 0} \left\{ \beta \lambda_n^* \left(\frac{1}{\beta}, \frac{m}{\beta} \right) \right\} \\ &= \inf_{\beta > 0} \sup_{x,y} \left\{ x + my - \beta \ln \mathbb{E}[\mathrm{e}^{x\tau + yW_n}] \right\}, \end{aligned}$$

(where $W_n = W \wedge n \vee (-n)$).

Since $-|W| \leq W_n \leq |W|$, we obtain by differentiating whether y is positive or negative that

$$x + my - \beta \ln \mathbb{E}[\mathrm{e}^{x\tau + yW_n}] \geq x + my - \beta \ln \mathbb{E}[\mathrm{e}^{x\tau + |y||W|}].$$

Therefore, we deduce the lower bound

$$J^n(m) \geq J^{|\cdot|}(m) := \inf_{\beta > 0} \sup_{x \in \mathbb{R}, y \geq 0} \left\{ x + |m| y - \beta \ln \mathbb{E}[\mathrm{e}^{x\tau + y|W|}] \right\}.$$

Remark that $J^{|\cdot|}$ is an even function, thus by symmetry, we can assume $m \geq 0$. Now, using Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned} &\sup_{x \in \mathbb{R}, y \geq 0} \left\{ x + my - \beta \ln \mathbb{E}[\mathrm{e}^{x\tau + y|W|}] \right\} \\ &\qquad = \sup_{x \in \mathbb{R}, y \geq 0} \left\{ x + my - \frac{\beta}{2} \ln \mathbb{E}[\mathrm{e}^{x\tau + y|W|}]^2 \right\} \\ &\qquad \geq \sup_{x \in \mathbb{R}, y \geq 0} \left\{ x + my - \frac{\beta}{2} \ln \mathbb{E}[\mathrm{e}^{2x\tau}] - \frac{\beta}{2} \ln \mathbb{E}[\mathrm{e}^{2y|W|}] \right\} \\ &\qquad \geq \sup_{x \in \mathbb{R}} \left\{ x - \frac{\beta}{2} \ln \mathbb{E}[\mathrm{e}^{2x\tau}] \right\} + \sup_{y \geq 0} \left\{ my - \frac{\beta}{2} \ln \mathbb{E}[\mathrm{e}^{2y|W|}] \right\} \end{aligned}$$

There exists $x_0 < 0$ such that $\mathbb{E}[\mathrm{e}^{2x_0\tau}] \leq \mathrm{e}^{-1}$, and since $\beta_0 > 0$, $y \mapsto \mathbb{E}[\mathrm{e}^{y|W|}]$ is strictly increasing and continuous on $[0, \beta_0)$. Then there exists $y_0 > 0$ such that

$1 < \mathbb{E}[\mathrm{e}^{2y_0|W|}] \leq \mathrm{e}$. Therefore

$$\sup_{x\in\mathbb{R},y\geq 0}\left\{x+my-\frac{\beta}{2}\ln\mathbb{E}[\mathrm{e}^{x\tau+y|W|}]^2\right\} \geq x_0-\frac{\beta}{2}\ln\mathbb{E}[\mathrm{e}^{2x_0\tau}]+my_0-\frac{\beta}{2}\ln\mathbb{E}[\mathrm{e}^{2y_0|W|}]$$
$$\geq x_0+my_0+\frac{\beta}{2}\left(1-\ln\mathbb{E}[\mathrm{e}^{2y_0|W|}]\right)$$

Finally, since $1-\ln\mathbb{E}[\mathrm{e}^{2y_0|W|}] > 0$, we conclude that

$$J^{|\cdot|}(m) \geq x_0+my_0 \underset{m\to\infty}{\longrightarrow} +\infty.$$

In conclusion, let $\alpha > 0$ and $\delta > \frac{2\alpha}{\beta_0}$ (or $\delta > 0$ if $\beta_0 = +\infty$), let $n(\alpha,\delta)$ defined in (2.9.2) and $n \geq n(\alpha,\delta)$, let $A = \frac{\alpha - x_0}{y_0} + \delta$ (x_0 and y_0 are determined by the law of τ and W). We have

$$\begin{aligned}\overline{\lim_t}\,\frac{1}{t}\ln\mathbb{P}\left(\frac{Z_t^n}{t} > A-\delta\right) &\leq -\inf_{m>A-\delta} J^n(m)\\ &\leq -\inf_{m>-A+\delta} J^{|\cdot|}(m)\\ &\leq -\inf_{m>A-\delta} x_0+my_0\\ &\leq -x_0-y_0(A-\delta)\\ &\leq -\alpha\end{aligned}$$

and using the evenness of $J^{|\cdot|}$,

$$\overline{\lim_t}\,\frac{1}{t}\ln\mathbb{P}\left(\frac{Z_t^n}{t} < -A+\delta\right) \leq -\alpha.$$

Eventually combining these bounds with (2.9.2) in (2.9.1), and deduce that for $n \geq n(\alpha,\delta)$,

$$\overline{\lim_t}\frac{1}{t}\ln\mathbb{P}\left(\frac{Z_t}{t} \notin [-A,A]\right) \qquad\qquad \leq -\alpha.$$

□

Now, we have a full LDP for Z_t/t, but the expression of $\tilde{J}$ isn't very convenient. In particular, this expression can depend on the reduction used. In order to simplify some inequality and obtain Equations (2.2.8) and (2.2.9), we prove $\tilde{J} \geq J$ defined in (2.2.6).
We consider two cases: in the first one, Lemma 2.9.2, we assume W is bounded by K and consider the discretization W^n defined in subsection 2.3.2. Then in the second one, Lemma 2.9.5, we will examine the general case.

Contrary to [52] we do not know a priori that $\tilde{J}$ is a good rate function. However, we use results of Lemma 2.8.1 replacing $\{1,...,n\}$ by $[-K,K]$. In particular we deduce that

$$J(m) = \bar{J}(m) := \min\{I(\mu)\,;\,\mu\in\mathcal{M}^1((0,+\infty]^2\times[-K,K])\,,\,\mu(\varphi)=m\}\,.$$

We want to prove

Lemma 2.9.2. If W is bounded, it holds $J \leq \tilde{J}$.

Proof. We may of course assume that $\tilde{J}(m) < +\infty$. One can thus find sequences m_n and $\varepsilon_n \leq 1$ such that $\varepsilon_n \to 0$, $m_n \to m$ and $J^n(m_n) \leq \tilde{J}(m) + \varepsilon_n$. Since $J^n = \bar{J}^n$, one can thus find a sequence $\mu_n \in \mathcal{M}^1((0,+\infty]^2 \times [-K,K])$ such that $I_n(\mu_n) < \tilde{J}(m) + 1$. According to proposition 2.5.3 (recall that it is true here, see remark 2.5.5) one can thus find some subsequence still denoted by μ_n converging to some μ in $\mathcal{M}^1((0,+\infty]^2 \times [-K,K])$. Now (again recall remark 2.5.5) according to proposition 2.5.4 we have $\liminf_n I_n(\mu_n) \geq I(\mu)$ so that $I(\mu) \leq \tilde{J}(m)$.

In order to see that $I(\mu) \geq J(m)$ it remains to show that $\mu(c/(a+b)) = m$ and to apply $J = \bar{J}$.

Notice that thanks to ii) in lemma 2.8.1 we may have chosen $\mu_n \in \Delta_0^n$ so that proposition 2.5.3 tells us in addition that $\limsup_n \mu_n(1/(a+b)) = M < +\infty$ (since the sequence $\alpha_n = 1$). It follows

$$\mu(1/(a+b+\varepsilon)) = \lim_n \mu_n(1/a+b+\varepsilon)) \leq \limsup_n \mu_n(1/(a+b)) = M$$

so that again using monotone convergence we deduce $\mu(1/(a+b)) \leq M$. Hence, since c is μ a.s. bounded, $\mu(c/(a+b))$ is well defined.

To calculate $\mu(c/(a+b))$ we need more. Using the definition of $\bar{J}^n$ and what precedes the sequence μ_n satisfies

$$H(\tilde{\pi}^n|\psi^n) = \frac{m_n}{\mu_n(1/(a+b))} \leq C$$

for some $C < +\infty$ since m_n is bounded and $\lim_n \mu_n(1/(a+b)) = \mu(1/(a+b)) > 0$. We will deduce that $\tilde{\pi}^n$ is tight. Indeed a standard application of the Orlicz-Hölder inequality (with the conjugate pair $u \mapsto e^u - 1 - u$ and $u \mapsto u \mapsto u \ln u - u$) shows that

$$\tilde{\pi}^n(A) \leq \kappa\, C\, \frac{1}{\ln(1/\psi^n(A))}$$

for some universal constant κ. Choosing $A = \{a+b < \varepsilon\}$ so that $\psi^n(A) = \psi(A)$ we get the desired result choosing ε small enough.

It follows

$$\begin{aligned}\left|\mu_n\left(\frac{c}{a+b}\,\left(\varepsilon^{-1}(a+b)1\!\!1_{a+b\leq\varepsilon} + 1\!\!1_{a+b>\varepsilon}\right)\right) - m_n\right| \leq &\quad K\,\mu_n(1/(a+b))\,C\kappa\,\mu_n(\{a+b\leq\varepsilon\}) \\ \leq &\qquad\qquad C'\,\mu(\{a+b\leq\varepsilon\})\,.\end{aligned}$$

Since the integrated function is bounded and continuous we can pass to the limit in n first, and then in ε using Lebesgue's bounded convergence theorem since

$$\left|\frac{c}{a+b}\,\left(\varepsilon^{-1}(a+b)1\!\!1_{a+b\leq\varepsilon} + 1\!\!1_{a+b>\varepsilon}\right)\right| \leq \frac{K}{a+b}$$

which is μ integrable according to what we did before. This finally shows that $\mu(c/(a+b)) = m$ and concludes the proof. $\square$

We do not know about the converse inequality.

Remark 2.9.3. We will see below another way to prove this result (in an even more general context). Nevertheless we have given this proof in order to complete the picture in the [52] context. ♢

We will now directly study the function J. Recall that

$$J(m) = \inf_{\beta>0} \sup_{x,y} \left(x + my - \beta \ln \mathbb{E}(e^{x\tau+yW})\right) := \inf_{\beta>0} \sup_{x,y} \Lambda(m,\beta,x,y)\,.$$

We do not assume here that W is bounded.

Lemma 2.9.4. J is a good rate function.

Proof. First we remark that

$$\sup_{x,y} \Lambda(m,\beta,x,y) \geq \sup_{x} \Lambda(m,\beta,x,0) = \sup_{x} \left(x - \beta \ln \mathbb{E}(e^{x\tau})\right)\,.$$

Since $\tau \geq 0$ (and supposed not to be identically 0) one can find $x_\tau < 0$ such that $\mathbb{E}(e^{x_\tau \tau}) = e^{-1}$ (we already use this in the proof of lemma 2.3.1), so that

$$\sup_{x,y} \Lambda(m,\beta,x,y) \geq x_\tau + \beta\,.$$

Let $\{J \leq M\}$ be some level set of J. For $\beta > M - x_\tau := \beta_\tau$ one has $\sup_{x,y} \Lambda(m,\beta,x,y) > M$ so that for all $m \in \{J \leq M\}$ it holds

$$J(m) = \inf_{0<\beta\leq\beta_\tau} \sup_{x,y} \Lambda(m,\beta,x,y)\,.$$

Now remark that

$$\sup_{x,y} \Lambda(m,\beta,x,y) \geq \sup_{y} \Lambda(m,\beta,0,y) \geq \Lambda(m,\beta,0,\kappa)$$

where $0 \leq \kappa < 1 \wedge \beta_0$, β_0 being defined in Assumption 2.2.4 *ii*). For $\beta \leq \beta_\tau$, $\beta \ln \mathbb{E}(e^{\kappa W})$ is thus bounded by C, so that

$$J(m) \geq m\kappa - C$$

showing that $\{J \leq M\}$ is bounded.

It remains to show that the level sets are closed. Let $m_n \to m$ be a sequence in $\{J \leq M\}$. According to what precedes we know that the infimum in β has to be taken in a bounded interval, so that one can find a (sub)-sequence β_n converging to β such that

$$J(m_n) \leq \sup_{x,y} \Lambda(m_n,\beta_n,x,y) + \varepsilon_n$$

with ε_n going to 0. This implies that for all (x,y)

$$x + m_n y - \beta_n \ln \mathbb{E}(e^{x\tau+yW}) \leq M + \varepsilon_n$$

which implies $J(m) \leq M$ by taking the limit in n and then the supremum w.r.t. (x,y). □

For a general W, consider

$$\tilde{J}(m) = \sup_{\delta>0} \liminf_{n\to\infty} \inf_{|m-z|<\delta} J^n(z)\,.$$

We now know that Z^n_t/t satisfies a full LDP with good rate function J^n. We may state

Lemma 2.9.5. It holds $J \le \tilde{J}$.

Proof. As for the proof of lemma 2.9.2, we may of assume that $\tilde{J}(m) < +\infty$ and thus find sequences m_n and $\varepsilon_n \le 1$ such that $\varepsilon_n \to 0$, $m_n \to m$ and $J^n(m_n) \le \tilde{J}(m)+\varepsilon_n$. We denote $\delta_n = |m - m_n|$. Define $M = \tilde{J}(m) + 1$. For n large enough, m_n belongs to $\{J^n \le M\}$. We have seen in the proof of lemma 2.9.4 that $J_n(m_n)$ is thus given by the infimum for $0 < \beta \le M - |x_\tau| = \beta_\tau$ where $\mathbb{E}(e^{x_\tau \tau}) = e^{-1}$, i.e x_τ does not depend on n. The same holds with $J(m)$. If $\beta \in (0, \beta_\tau]$ and $\varepsilon > 0$ we may found $(x_\varepsilon, y_\varepsilon)$ such that

$$\begin{aligned}
\beta\Lambda^*(1/\beta, m/\beta) \;&\le\; x_\varepsilon + m y_\varepsilon - \beta\,\ln\left(\mathbb{E}\left[e^{x_\varepsilon\tau+y_\varepsilon W}\right]\right) + \varepsilon \\
&\le\; x_\varepsilon + m_n y_\varepsilon - \beta\,\ln\left(\mathbb{E}\left[e^{x_\varepsilon\tau+y_\varepsilon W}\right]\right) + \varepsilon + \delta_n|y_\varepsilon|\,.
\end{aligned}$$

Since

$$\mathbb{E}\left[e^{x_\varepsilon\tau+y_\varepsilon W^n}\right] \to \mathbb{E}\left[e^{x_\varepsilon\tau+y_\varepsilon W}\right]$$

as n growths to infinity, for n large enough the difference is less than ε so that

$$\begin{aligned}
\beta\Lambda^*(1/\beta, m/\beta) \;&\le\; x_\varepsilon + m_n y_\varepsilon - \beta\,\ln\left(\mathbb{E}\left[e^{x_\varepsilon\tau+y_\varepsilon W^n}\right]\right) + (1+\beta_\tau)\varepsilon + \delta_n|y_\varepsilon| \\
&\le\; \beta\Lambda^*_n(1/\beta, m_n/\beta) + (1+\beta_\tau)\varepsilon + \delta_n|y_\varepsilon|\,.
\end{aligned}$$

Taking the infimum in β and then the $\liminf_n$ we get

$$J(m) \le \liminf_n J_n(m_n) + (1+\beta_\tau)\varepsilon \;\le\; \tilde{J}(m) + (1+\beta_\tau)\varepsilon.$$

It remains to let ε go to 0 to conclude. □

Remark 2.9.6. When $W = 1$ a.s. it is easily seen that $\sup_{x,y} \Lambda(m, \beta, x, y) = +\infty$ except for $m = \beta$ yielding $J(m) = \sup_x\,(x - m\ln\mathbb{E}(e^{x\tau}))$ as expected.

Since J is defined on $\mathbb{R}$ one can expect some monotonicity on intervals delimited by the asymptotic mean $\mathbb{E}(W)/\mathbb{E}(\tau)$. We were not able to prove this monotonicity. ♢

2.9.2 Case B: $\beta_0 < \infty$

We remind that W^n is a reduction of W such that Z^n_t/t is an approximation of Z_t/t and satisfies a full LDP, as proved in Theorem 2.4.4. However, this approximation is not an exponentially good approximation, therefore the LDP cannot be transferred to Z_t/t. In this case, we prove the deviation Inequalities (2.2.10) and (2.2.11). We will focus on the first one, $\mathbb{P}\left(\frac{Z_t}{t} \ge m + a\right)$, since the proof is exactly the same for the second one, $\mathbb{P}\left(\frac{Z_t}{t} \leq m-a\right)$.

For each $n \in \mathbb{N}^*$, for $\kappa \in (0,1)$, we have

$$\begin{aligned}\mathbb{P}\left(\frac{Z_t}{t} \geq m+a\right) &\leq \mathbb{P}\left(\frac{Z_t^n}{t} \geq m+\kappa a\right) + \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{Z_t^n}{t}\right| \geq (1-\kappa)a\right)\\ &\leq 2\max\left[\mathbb{P}\left(\frac{Z_t^n}{t} \geq m+\kappa a\right), \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{Z_t^n}{t}\right| \geq (1-\kappa)a\right)\right]\end{aligned}$$

Then, for each $n \in \mathbb{N}^*$ and $\kappa \in (0,1)$, we obtain

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\ln\mathbb{P}\left(\frac{Z_t}{t} \geq m+a\right)\\ &\qquad \leq \max\left[\limsup_{t\to\infty}\frac{1}{t}\ln\mathbb{P}\left(\frac{Z_t^n}{t} \geq m+\kappa a\right), \limsup_{t\to\infty}\frac{1}{t}\ln\mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{Z_t^n}{t}\right| \geq (1-\kappa)a\right)\right].\end{aligned}$$

We handle the first term with the full LDP for Z_t^n/t with the rate function J^n (Theorem 2.4.4) and the second term with Lemma 2.3.1 (or 2.3.2, or 2.3.4 depending on the approximation strategy). We then have

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\ln\mathbb{P}\left(\frac{Z_t}{t} \geq m+a\right)\\ &\qquad \leq \max\left[\liminf_{n\to\infty}\left(-\inf_{z\geq m+\kappa a} J^n(z)\right), -\beta_0\frac{(1-\kappa)a}{2}\right]\\ &\qquad \leq -\min\left[\limsup_{n\to\infty}\inf_{z\geq m+\kappa a} J^n(z), \beta_0\frac{(1-\kappa)a}{2}\right].\end{aligned}$$

Since this inequality is satisfied for each n, we can then apply the Lemma 2.9.8, proved below, which gives us

$$\limsup_{n\to\infty}\inf_{z\geq m+\kappa a} J^n(z) \geq \inf_{z\geq m+\kappa a} J(z),$$

and we obtain the Inequation (2.2.10): for all $\kappa \in (0,1)$

$$\limsup_{t\to\infty}\frac{1}{t}\ln\mathbb{P}\left(\frac{Z_t}{t} \geq m+a\right) \leq -\min\left[\inf_{z\geq m+\kappa a} J(z), \beta_0\frac{(1-\kappa)a}{2}\right].$$

Remark 2.9.7. Notice J^n isn't the same in each case. J^n can be written

$$J^n(z) = \inf_{\beta>0}\sup_{x,y}\{x+my-\beta\ln\mathbb{E}\left(e^{x\tau+yW^n}\right)\},$$

but the definition of W^n depends on the reduction:

— If W is only bounded by K, a reduction with finite valued W^n was necessary:

$$W^n = \sum_{j=-n}^{n-1}\frac{jK}{n}\mathbb{1}_{W\in[jK/n,(j+1)K/n[}.$$

— If W is only discrete, a reduction with W^n bounded by $-n$ and n have been done: $W^n = (W \vee n) \wedge (-n)$.

— If W isn't bounded nor discrete, another reduction is necessary:

$$\tilde{W}^n = -n\mathbb{1}_{W<-n} + n\mathbb{1}_{W\geq n} + \sum_{j=-n^2}^{n^2-1} \frac{j}{n}\mathbb{1}_{W\in[\frac{j}{n},\frac{j+1}{n})}.$$

Since in each case, W^n is discrete and bounded a.s. and $E\left(e^{x\tau+yW^n}\right) \to_{n\to\infty} \mathbb{E}\left(e^{x\tau+yW}\right)$ for (x,y) such that $\mathbb{E}\left(e^{x\tau+yW}\right) < \infty$, the proof is the same.

Lemma 2.9.8. Assume $(W^n)_n$ is a sequence of random variables which converges almost surely to W and such that for each n, W^n is discrete and bounded a.s. Then, we have, for all $z_0 \in \mathbb{R}$,

$$\limsup_{n\to\infty} \inf_{z\geq z_0} J^n(z) \geq \inf_{z\geq z_0} J(z),$$

where J^n is defined by

$$J^n(z) = \inf_{\beta>0} \sup_{x,y}\{x + my - \beta \ln \mathbb{E}\left(e^{x\tau+yW^n}\right)\}.$$

Proof. Let $(W^n)_n$ be a sequence of random variables which converges almost surely to W, such that for each n, W^n is discrete and bounded a.s. Let $z_0 \in \mathbb{R}$. Thanks to Lemma 2.8.1, J^n has different forms. Here we will use the form

$$J^n(z) = \inf_{\beta>0} \beta\Lambda_n^*\left(\frac{1}{\beta}, \frac{m}{\beta}\right).$$

If $\limsup_{n\to\infty} \inf_{z\geq z_0} J^n(z) = \infty$, then the inequality is satisfied.
Let assume $\limsup_{n\to\infty} \inf_{z\geq z_0} J^n(z) < \infty$. We denote $C(z_0)$ this quantity.

There exists a sequence $n_k \to \infty$ such that $\lim_{k\to\infty} \inf_{z\geq z_0} J^{n_k}(z) = C(z_0)$. For each $k \in \mathbb{N}$, there exists a sequence $(z_i^k)_i$ in $[z_0, +\infty)$ such that $\inf_{z\geq z_0} J^{n_k}(z) = \lim_i J^{n_k}(z_i^k)$.
Let $\delta > 0$. There exists K such that $\forall k \geq K$, $\inf_{z\geq z_0} J^{n_k}(z) \in [C(z_0)-\delta/2, C(z_0)+\delta/2]$ and I_k such that $\forall i \geq I_K$, $J^{n_k}(z_i^k) \in [C(z_0)-\delta, C(z_0)+\delta]$.
Then, for $k \geq K$, and $i \geq I_k$, z_i^k is in $\{J^{n_k} \leq C(z_0)+\delta\}$.
Moreover, for each $n \in N$, $J^n \geq J^{|\cdot|}$ by the proof of the Lemma 2.9.1, where

$$J^{|\cdot|}(m) := \inf_{\beta>0} \sup_{x\in\mathbb{R},y\geq 0} \left\{x + |m|y - \beta \ln \mathbb{E}[\mathrm{e}^{x\tau+y|W|}]\right\}.$$

So for $k \geq K$, and $i \geq I_k$, z_i^k is in $\{J^{|\cdot|} \leq C(z_0)+\delta\}$.
By the exact same argument than for J in the proof of Lemma 2.9.4, $\{J^{|\cdot|} \leq C(z_0)+\delta\}$ is a compact level set of $J^{|\cdot|}$.
Then, for $k \geq K$ and $i \geq I_k$, z_i^k is in a compact. There exists at least an adherent point $z_{\lim}$ in this compact, and a subsequence $z_{i_j}^{k_j} \to_{k_j,i_j\to\infty} z_{\lim}$. In particular, $z_{\lim} \geq z_0$.

Like in the proof of Lemma 2.9.5, $J^{n_{k_j}}(z_{i_j}^{k_j})$ is given by the infimum for $\beta \in (0, \beta_\tau)$. If $\beta \in (0, \beta_\tau)$ and $\varepsilon > 0$, we may found $(x_\varepsilon, y_\varepsilon)$ such that

$$\begin{aligned}\beta\Lambda^*(1/\beta, z_{\lim}/\beta) &\leq x_\varepsilon + my_\varepsilon - \beta \ln\left(\mathbb{E}\left[e^{x_\varepsilon \tau + y_\varepsilon W}\right]\right) + \varepsilon \\ &\leq x_\varepsilon + z_{i_j}^{k_j} y_\varepsilon - \beta \ln\left(\mathbb{E}\left[e^{x_\varepsilon \tau + y_\varepsilon W}\right]\right) + \varepsilon + |z_{\lim} - z_{i_j}^{k_j}||y_\varepsilon|\end{aligned}$$

Since

$$\mathbb{E}\left[e^{x_\varepsilon \tau + y_\varepsilon W^n}\right] \to \mathbb{E}\left[e^{x_\varepsilon \tau + y_\varepsilon W}\right]$$

as n growths to infinity, for n large enough the difference is less than ε so that

$$\begin{aligned}\beta\Lambda^*(1/\beta, z_{\lim}/\beta) &\leq x_\varepsilon + z_{i_j}^{k_j} y_\varepsilon - \beta \ln\left(\mathbb{E}\left[e^{x_\varepsilon \tau + y_\varepsilon W^n}\right]\right) + (1+\beta_\tau)\varepsilon + |z_{\lim} - z_{i_j}^{k_j}||y_\varepsilon| \\ &\leq \beta\Lambda_n^*(1/\beta, z_{i_j}^{k_j}/\beta) + (1+\beta_\tau)\varepsilon + |z_{\lim} - z_{i_j}^{k_j}||y_\varepsilon|\end{aligned}$$

Taking the infimum in β and then the $\lim_j$ we get

$$J(z_{\lim}) \leq \liminf_j J^{n_{k_j}}(z_{i_j}^{k_j}) + (1+\beta_\tau)\varepsilon \leq \limsup_{n\to\infty} \inf_{z \geq z_0} J^n(z) + (1+\beta_\tau)\varepsilon.$$

By letting ε go to 0 we obtain:

$$J(z_{\lim}) \leq \limsup_{n\to\infty} \inf_{z \geq z_0} J^n(z).$$

Then,

$$\inf_{z \geq z_0} J(z) \leq J(z_{\lim}) \leq \limsup_{n\to\infty} \inf_{z \geq z_0} J^n(z).$$

□

2.A Lemmata and Propositions from [52]

The following Lemmata and Propositions are from [52]. Some proofs are exactly the same, and we give them to have the totality of the arguments in this manuscript. Some are slightly different from the original article, since we have a third coordinate: it will be indicated in the title of the Lemmata and Propositions. We can have developed some of the proofs.

2.A.1 Complement to Section 2.5 *The functional* I^n

The following lemma is a rewriting of $\pi(1/\tau)H(\tilde{\pi}|\psi^n)$ which is useful to work on the rate function I^n.

Lemma 2.A.1 (Lemma 2.2 from [52]). For all $\pi \in \mathcal{M}^1((0,+\infty)\times\{1,...,n\})$ such that $\pi(1/\tau) < \infty$ and $a > 0$

$$\begin{aligned}\pi(1/\tau)H(\tilde{\pi}|\psi^n) &= \sup_{\varphi\in\mathcal{C}_b((0,+\infty)\times\{1,...,n\})}\left(\pi(\varphi/\tau) - \pi(1/\tau)\ln\psi^n\left(\mathrm{e}^{\varphi}\right)\right)\\ &= \sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})=a}\left(\pi(\varphi/\tau) - \pi(1/\tau)\ln\psi^n\left(\mathrm{e}^{\varphi}\right)\right.\\ &= \sup_{f\in\mathcal{C}((0,+\infty)\times\{1,...,n\}),\pi(f)<\infty}\left(\pi(f) - \pi(1/\tau)\ln\psi^n(\mathrm{e}^{\tau f})\right).\end{aligned} \tag{2.A.1}$$

Proof. We know that

$$H(\tilde{\pi}|\psi^n) = \sup_{\varphi\in\mathcal{C}_b((0,+\infty)\times\{1,...,n\})}\left(\tilde{\pi}(\varphi) - \ln\psi^n(\mathrm{e}^{\varphi})\right).$$

Let φ such a function, and $a = \psi^n(\mathrm{e}^{\varphi})$. Suppose $a > 0$, set $\varphi_a := \varphi - \ln a$. Then

$$\begin{aligned}\pi(1/\tau)\times\left(\tilde{\pi}(\varphi) - \ln\psi^n(\mathrm{e}^{\varphi})\right) &= \pi(1/\tau)\times\frac{\pi\left(\varphi/\tau\right)}{\pi(1/\tau)} - \pi(1/\tau)\ln\psi^n(\mathrm{e}^{\varphi})\\ &= \pi\left(\varphi/\tau\right) - \pi(1/\tau)\ln\psi^n(\mathrm{e}^{\varphi})\\ &= \pi\left(\varphi_a/\tau\right) + \ln(a)\pi\left(1/\tau\right) - \pi(1/\tau)\ln\left(\psi^n(\mathrm{e}^{\varphi_a})\times a\right)\\ &= \pi\left(\varphi_a/\tau\right) - \pi(1/\tau)\ln\psi^n(\mathrm{e}^{\varphi_a}).\end{aligned}$$

Moreover $\psi^n(\mathrm{e}^{\varphi_a}) = \psi^n(\mathrm{e}^{\varphi})\times a^{-1} = 1$. Then the quantity:

$$\sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})=a}\left(\pi(\varphi/\tau) - \pi(1/\tau)\ln\psi^n\left(\mathrm{e}^{\varphi}\right)\right) = \sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})=1}\left(\pi(\varphi/\tau) - \pi(1/\tau)\ln\psi^n\left(\mathrm{e}^{\varphi}\right)\right)$$

and doesn't depends on $a > 0$. Therefore

$$\begin{aligned}\pi(1/\tau)H(\tilde{\pi}|\psi^n) &= \sup_{\varphi\in\mathcal{C}_b((0,+\infty)\times\{1,...,n\})}\left(\pi(\varphi/\tau) - \pi(1/\tau)\ln\psi^n\left(\mathrm{e}^{\varphi}\right)\right)\\ &= \sup_{a}\sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})=a}\left(\pi(\varphi/\tau) - \pi(1/\tau)\ln\psi^n\left(\mathrm{e}^{\varphi}\right)\right.\\ &= \sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})=a}\left(\pi(\varphi/\tau) - \pi(1/\tau)\ln\psi^n\left(\mathrm{e}^{\varphi}\right)\right..\end{aligned}$$

For the last equality, we consider $\varphi = \tau f$.
First, for a given $\varphi \in \mathcal{C}_b((0,+\infty)\times\{1,...,n\})$, $f(\tau, W) := \dfrac{\varphi(\tau, W)}{\tau}$ satisfies $\pi(f) < \infty$ because:

$$\pi(f) = \pi\left(\frac{\varphi}{\tau}\right) \leq \|\varphi\|_\infty \pi(1/\tau) < \infty.$$

Then:

$$\sup_{\varphi\in\mathcal{C}_b((0,+\infty)\times\{1,...,n\})} \left(\pi(\varphi/\tau) - \pi(1/\tau)\ln\psi^n\left(\mathrm{e}^{\varphi}\right)\right) \quad \leq \sup_{\substack{f\in\mathcal{C}((0,+\infty)\times\{1,...,n\})\\ \pi(f)<\infty}} (\pi(f) - \pi(1/\tau)\ln\psi^n(\mathrm{e}^{\tau f})).$$

Secondly, we can restrict the supremum on f to functions such that $\psi^n(\mathrm{e}^{\tau f}) < \infty$. We consider $\chi_M \in \mathcal{C}_b((0,+\infty)\times\{1,...,n\})$ such that $\chi_M \in [0,1]$, $\chi_M(\tau) = 0$ for $\tau < M^{-1}$ and $\tau > 2M$ and $\chi_M(\tau) = 1$ for $\tau \in [2M^{-1}, M]$.
Then, for $f \in \mathcal{C}((0,+\infty)\times\{1,...,n\})$ such that $\pi(f) < +\infty$ and $\psi^n(\mathrm{e}^{\tau f}) < +\infty$, let $\varphi_M = \tau f \chi_M$. φ_M is in $\mathcal{C}_b((0,+\infty)\times\{1,...,n\})$. By dominated convergence, we have:

$$\lim_{M\to\infty} \pi(\varphi_M/\tau) - \pi(1/\tau)\ln\psi^n(\mathrm{e}^{\varphi_M}) = \pi(f) - \pi(1/\tau)\ln\psi^n(\mathrm{e}^{\tau f}).$$

The equality is then established. □

Lemma 2.A.2 (Lemma 2.3 from [52] - slightly different proof). The map $\pi \mapsto \pi(1/\tau)H(\tilde{\pi}|\psi^n)$ is convex on $\{\pi \in \mathcal{M}^1((0,+\infty)\times\{1,...,n\}), \pi(1/\tau) < \infty\}$. Moreover I^n is also convex.

Proof. By the lemma 2.A.1, $\pi \mapsto \pi(1/\tau)H(\tilde{\pi}|\psi^n)$ can be expressed as a supremum of linear functions of π, then this map is convex.
To prove the convexity of I^n, let μ_1 and μ_2 in Δ^n, $\beta \in (0,1)$ and $\mu = \beta\mu_1 + (1-\beta)\mu_2$.
We want to prove $I^n(\mu) \leq \beta I^n(\mu_1) + (1-\beta)I^n(\mu_2)$.
This inequality is true if μ_1 or μ_2 aren't in Δ^n.
Let suppose μ is in Δ^n. We denote by $\alpha_1, \alpha_2, \pi_1$, π_2, η_1 and η_2 the corresponding quantities of the equation 2.4.2 for μ_1 and μ_2.

If $\alpha_1 = \alpha_2 = 0$, then $\mu = \beta\mu_1 + (1-\beta)\mu_2 = \delta_{(+\infty,+\infty)} \otimes (\beta\eta_1 + (1-\beta)\eta_2)$, $I^n(\mu) = \xi = \beta\xi + (1-\beta)\xi$ and the inequality is satisfied.

Else, we have for μ:

$$\begin{aligned}
\alpha &= \beta\alpha_1 + (1-\beta)\alpha_2 \\
\pi &= \frac{\beta\alpha_1}{\alpha}\pi_1 + \frac{(1-\beta)\alpha_2}{\alpha}\pi_2 \\
\eta &= \frac{\beta(1-\alpha_1)}{1-\alpha}\eta_1 + \frac{(1-\beta)(1-\alpha_2)}{1-\alpha}\eta_2.
\end{aligned}$$

We notice that $1-\alpha = (\beta+(1-\beta))-(\beta\alpha_1+(1-\beta)\alpha_2) = \beta(1-\alpha_1)+(1-\beta)(1-\alpha_2)$. Therefore, by the convexity of $\pi \mapsto \pi(1/\tau)H(\tilde{\pi}|\psi^n)$,

$$\begin{aligned}
I^n(\mu) &= \alpha\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1-\alpha)\xi \\
&\leq \alpha\left[\frac{\beta\alpha_1}{\alpha}\pi_1(1/\tau)H(\tilde{\pi_1}|\psi^n) + \frac{(1-\beta)\alpha_2}{\alpha}\pi_2(1/\tau)H(\tilde{\pi_2}|\psi^n)\right] + (1-\alpha)\xi \\
&\leq \beta\alpha_1\pi_1(1/\tau)H(\tilde{\pi_1}|\psi^n) + (1-\beta)\alpha_2\pi_2(1/\tau)H(\tilde{\pi_2}|\psi^n) \\
&\quad + \xi\left(\beta(1-\alpha_1) + (1-\beta)(1-\alpha_2)\right) \\
&\leq \beta I^n(\mu_1) + (1-\beta)I^n(\mu_2).
\end{aligned}$$

Then I^n is convex. □

Lemma 2.A.3 (Lemma 2.4 from [52]). Let $\pi_k \in \mathcal{M}^1((0,+\infty)\times\{1,...,n\})$ be such that $\pi_k(1/\tau) < \infty$ such that

$$\lim_k \pi_k(d\tau, dW) = \beta\pi(d\tau, dW) + (1-\beta)\delta_{(+\infty)}(d\tau)\otimes\eta_0(dW) \tag{2.A.2}$$

for some $\beta \in [0,1]$, $\pi \in \mathcal{M}^1((0,+\infty)\times\{1,...,n\})$ such that $\pi(1/\tau) < \infty$ and $\eta_0 \in \mathcal{M}^1(\{1,...,n\})$. Then

$$\liminf_{k\to\infty} \pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k) \geq \beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1-\beta)\xi.$$

Proof. By the lemma 2.A.1

$$\pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k) = \sup_{\substack{f\in\mathcal{C}((0,+\infty)\times\{1,...,n\}),\\ \pi_k(f)<\infty}} \left(\pi_k(f) - \pi_k(1/\tau)\ln\psi_k(\mathrm{e}^{\tau f})\right). \tag{2.A.3}$$

To prove this lemma, we define an adequate function f from some function φ. By some optimizations over φ and by the equations (2.A.1) from the lemma 2.A.1 we conclude.

We fix $\varphi \in \mathcal{C}_b((0,+\infty)\times\{1,...,n\})$ such that $\psi(\mathrm{e}^{\varphi}) < 1$. If $\xi = 0$, we consider $d = 0$, else we consider $d \in [0,\xi)$. For $M > 0$, let χ_M be a smooth function on $(0,+\infty)$ such that

$$\begin{aligned}
&\chi_M(\tau) = 1 \text{ for } \tau \leq \frac{1}{M+1} \text{ or } \tau \geq M+1 \\
&\chi_M(\tau) = 0 \text{ for } \tau \in \left[\frac{1}{M}, M\right].
\end{aligned}$$

We have

$$
\begin{aligned}
&\psi^n(\mathrm{e}^{d\tau\chi_M+\varphi(1-\chi_M)})\\
&\quad=\int_{(0,\frac{1}{M})\times\{1,\dots,n\}}\mathrm{e}^{d\tau\chi_M(\tau)+\varphi(\tau,W)(1-\chi_M(\tau))}\,\psi^n(d\tau,dW)+\int_{[\frac{1}{M},M]\times\{1,\dots,n\}}\mathrm{e}^{\varphi(\tau,W)}\,\psi^n(d\tau,dW)\\
&\qquad+\int_{(M+1,+\infty)\times\{1,\dots,n\}}\mathrm{e}^{d\tau\chi_M(\tau)+\varphi(\tau,W)(1-\chi_M(\tau))}\,\psi^n(d\tau,dW)\\
&\quad\leq \mathrm{e}^{\|\varphi\|_\infty}\underbrace{\psi^n\left(\mathrm{e}^{d\tau}\,\mathbb{1}_{\tau<\frac{1}{M}}\right)}_{\underset{M\to\infty}{\longrightarrow}0}+\psi^n(\mathrm{e}^{\varphi})+\mathrm{e}^{\|\varphi\|_\infty}\underbrace{\psi^n(\mathrm{e}^{d\tau}\,\mathbb{1}_{\tau>M})}_{\underset{M\to\infty}{\longrightarrow}0}\\
&\quad\underset{M\to\infty}{\longrightarrow}\psi^n(\mathrm{e}^{\varphi})<1.
\end{aligned}
$$

Then there exists M' which depends on φ and d such that $\forall M>M', \psi^n(\mathrm{e}^{d\tau\chi_M+\varphi(1-\chi_M)})<1$. (The fact that $\psi^n(\mathrm{e}^{d\tau})$ is finite implies the convergence of $\psi^n(\mathrm{e}^{d\tau}\,\mathbb{1}_{\tau>M})$ and $\psi^n\left(\mathrm{e}^{d\tau}\,\mathbb{1}_{\tau<\frac{1}{M}}\right)$ to 0.) Since $\psi_k\rightharpoonup\psi^n$, $\xi_k\to\xi>c$, we have for k large enough (depending on M',φ and d):

$$\forall M>M', \forall k \text{ large enough}, \psi_k(\mathrm{e}^{d\tau\chi_M+\varphi(1-\chi_M)})<1.$$

Let f be of the form $f(\tau,W)=d\chi_M(\tau)+\frac{\varphi(\tau,W)}{\tau}(1-\chi_M(\tau))$ and we apply (2.A.3) with it:

$$
\begin{aligned}
\pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k)&\geq\pi_k(f)-\pi_k(1/\tau)\ln\psi_k(\mathrm{e}^{\tau f})\\
&\geq\pi_k(d\chi_M(\tau))+\pi_k\left(\frac{\varphi(\tau,W)}{\tau}(1-\chi_M(\tau))\right)\underbrace{-\pi_k(1/\tau)\ln\psi_k(\mathrm{e}^{\tau f})}_{>0}\\
&\geq\pi_k(d\chi_M(\tau))+\pi_k\left(\frac{\varphi(\tau,W)}{\tau}(1-\chi_M(\tau))\right).
\end{aligned}
$$

Then, by (2.5.2) the statement of the lemma

$$
\begin{aligned}
&\liminf_{k\to\infty}\pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k)\\
&\qquad\geq\liminf_{k\to\infty}\pi_k(d\chi_M(\tau))+\pi_k\left(\frac{\varphi(\tau,W)}{\tau}(1-\chi_M(\tau))\right)\\
&\qquad\geq\beta\pi(d\chi_M(\tau))+(1-\beta)d\chi_M(+\infty)+\beta\pi\left(\frac{\varphi(\tau,W)}{\tau}(1-\chi_M(\tau))\right)+(1-\beta)\times 0\\
&\qquad\geq\beta d\pi(\chi_M(\tau))+(1-\beta)d+\beta\pi\left(\frac{\varphi(\tau,W)}{\tau}(1-\chi_M(\tau))\right)
\end{aligned}
$$

We take the limit when M tends to $+\infty$, $\chi_M\to 0$ point-wisely. Since $\pi(1/\tau)<\infty$ and $\pi(\tau=+\infty)=0$, we obtain by dominated convergence:

$$\beta d\pi(\chi_M(\tau))+(1-\beta)d+\beta\pi\left(\frac{\varphi(\tau,W)}{\tau}(1-\chi_M(\tau))\right)\underset{M\to+\infty}{\longrightarrow}(1-\beta)d+\beta\pi\left(\frac{\varphi(\tau,W)}{\tau}\right)$$

$$\liminf_{k\to\infty}\pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k)\geq(1-\beta)d+\beta\pi\left(\frac{\varphi(\tau,W)}{\tau}\right).$$

We optimize over $d < \xi$ and φ such that $\psi^n(\mathrm{e}^{\varphi}) < 1$:

$$\begin{aligned}\liminf_{k\to\infty} \pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k) &\geq \sup_{d<\xi}\sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})<1}\left((1-\beta)d + \beta\pi\left(\frac{\varphi(\tau,W)}{\tau}\right)\right)\\ &\geq (1-\beta)\xi + \beta\sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})<1}\pi\left(\frac{\varphi(\tau,W)}{\tau}\right).\end{aligned}$$

By the lemma 2.A.1:

$$\begin{aligned}\sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})<1}\pi\left(\frac{\varphi(\tau,W)}{\tau}\right) &= \sup_{a<1}\sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})=a}\pi\left(\frac{\varphi(\tau,W)}{\tau}\right)\\ &= \sup_{a<1}\left[\sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})=a}\left(\pi\left(\frac{\varphi(\tau,W)}{\tau}\right) - \ln\psi^n(\mathrm{e}^{\varphi}) + \ln\psi^n(\mathrm{e}^{\varphi})\right)\right]\\ &= \sup_{a<1}\left[\sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})=a}\left(\pi\left(\frac{\varphi(\tau,W)}{\tau}\right) - \ln\psi^n(\mathrm{e}^{\varphi})\right) + \ln a\right]\\ &= \sup_{a<1}\left[\pi(1/\tau)H(\tilde{\pi}|\psi^n) + \ln a\right]\\ &= \sup_{a<1}\ln a + \pi(1/\tau)H(\tilde{\pi}|\psi^n) = \pi(1/\tau)H(\tilde{\pi}|\psi^n).\end{aligned}$$

Then we obtain the announced inequality:

$$\liminf_{k\to\infty}\pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k) \geq \beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1-\beta)\xi.$$

□

2.A.2 Complement to Section 2.6 *Some additional technical topological lemmata.*

Lemma 2.A.4 (Decomposition of $\mu_t^n(f)$ - Equation (3.3) from [52]).
For $f : (0,+\infty]^2 \times \{1,...,n\}) \to \mathbb{R}$, bounded or positive, we have:

$$\begin{aligned}\mu_t^n(f) =& \frac{1}{t}\sum_{i=1}^{M_t}\tau_i\int_0^1 f(u\tau_i,(1-u)\tau_i,W_i^n)du\\ &+ \frac{\tau_{M_t+1}}{t}\int_0^{\frac{t-S_{M_t}}{\tau_{M_t+1}}} f(\tau_{M_t+1}u,(1-u)\tau_{M_t+1},W_{M_t+1}^n)du \end{aligned}\tag{2.A.4}$$

which can be expressed by:

$$\mu_t^n(f) = \frac{1}{t}\sum_{i=1}^{M_t}\tau_i\bar{f}(1,\tau_i,W_i^n) + \frac{\tau_{M_t+1}}{t}\bar{f}\left(\frac{t-S_{M_t}}{\tau_{M_t+1}},\tau_{M_t+1},W_{M_t+1}^n\right).\tag{2.A.5}$$

Proof. We have:

$$\begin{aligned}\mu_t^n(f) =&\frac{1}{t}\sum_{i=1}^{M_t}\int_{S_{i-1}}^{S_i} f(s-S_{M_s}, S_{M_s+1}-s, W^n_{M_s+1})ds\\&+\frac{1}{t}\int_{S_{M_t}}^{t} f(s-S_{M_s}, S_{M_s+1}-s, W^n_{M_s+1})ds\\=&\frac{1}{t}\sum_{i=1}^{M_t}\int_0^1 f(u\tau_i,(1-u)\tau_i,W_i^n)\tau_i du\\&+\frac{1}{t}\int_0^{\frac{t-S_{M_t}}{\tau_{M_t+1}}} f(\tau_{M_t+1}u,(1-u)\tau_{M_t+1},W^n_{M_t+1})\tau_{M_t+1}du,\end{aligned}$$

with $u=\dfrac{s-S_{i-1}}{\tau_i}$ for each i.

We recall (2.4.6): for all x, τ, W, $\bar{f}(x,\tau,w)=\int_0^x f(u\tau,(1-u)\tau,w)du$. We have:

$$\mu_t^n(f)=\frac{1}{t}\sum_{i=1}^{M_t}\tau_i\bar{f}(1,\tau_i,W_i^n)+\frac{\tau_{M_t+1}}{t}\bar{f}\left(\frac{t-S_{M_t}}{\tau_{M_t+1}},\tau_{M_t+1},W^n_{M_t+1}\right).$$

□

Lemma 2.A.5 (Lemma 2.6 from [52]). For all $\mu\in\Delta^n$, $I^n(\mu)\leq\sup_{f\in\Gamma}\mu(f)$.

Proof. The objective of this proof is to work with a set of functions f, such that f is in Γ and such that $\mu(f)$ is easily calculable. Then, by optimizing over the parameters of f, we prove the wished inequality.

Let $M>0$, $\gamma<1$ and $\varphi\in\mathcal{C}_c((0,+\infty]\times\{1,...,n\})$ such that $\psi^n(\mathrm{e}^{\varphi})=\gamma$. If $\xi=0$, let $d=0$, else let $d\in(0,\xi)$. Let

$$f_{d,\varphi,M}(a,b,c)=\frac{\varphi(a+b,c)}{a+b}+d\mathbb{1}_{(M,+\infty]}(a+b),(a,b,c)\in(0,+\infty]^2\times\{1,...,n\}.$$

First, we prove that $f_{d,\varphi,M}$ is in Γ.
$f_{d,\varphi,M}$ is lower semicontinuous on $(0,+\infty]^2\times\{1,...,n\}$. In fact, $g:x\mapsto g(x_0)$ is lower semicontinuous in x_0 if $\liminf_{x\to x_0}g(x)\geq g(x)$.
On $\{(a,b,c),a+b<M\}$, $f_{d,\varphi,M}(a,b,c)=\frac{\varphi(a+b,c)}{a+b}$ which is a continuous function (so lower semicontinuous). On $\{(a,b,c),a+b>M\}$, $f_{d,\varphi,M}(a,b,c)=\frac{\varphi(a+b,c)}{a+b}+d$ which is a continuous function, so lower semicontinuous. Let $(a_0,b_0,c_0)\in(0,+\infty]^2\times\{1,...,n\}$ such that $a_0+b_0=M$, and let $(a_k,b_k,c_k)\to(a_0,b_0,c_0)$. Then

$$\begin{aligned}f_{d,\varphi,M}(a_k,b_k,c_k)&=\frac{\varphi(a_k+b_k,c_k)}{a_k+b_k}+d\mathbb{1}_{(M,+\infty]}(a_k+b_k)\\&\geq\frac{\varphi(a_k+b_k,c_k)}{a_k+b_k}\ \text{(because } d\geq 0)\\\liminf_{k\to\infty}f_{d,\varphi,M}(a_k,b_k,c_k)&\geq\frac{\varphi(a_0+b_0,c_0)}{a_0+b_0}=f_{d,\varphi,M}(a_0,b_0,c_0).\end{aligned}$$

We can study $\bar{f}_{d,\varphi,M}$, defined in (2.4.6), for $r \in [0,1]$ and $\tau > 0$:

$$\begin{aligned}
\bar{f}_{d,\varphi,M}(r,\tau,c) &= \int_0^r f_{d,\varphi,M}(u\tau,(1-u)\tau,c)du \\
&= \int_0^r \frac{\varphi(\tau,c)}{\tau} + d\mathbb{1}_{(M,+\infty]}(\tau)du \\
&= r \times \left(\frac{\varphi(\tau,c)}{\tau} + d\mathbb{1}_{(M,+\infty]}(\tau)\right).
\end{aligned}$$

Then, let $s > 0$ and

$$\begin{aligned}
\int_{[s,+\infty)} \mathrm{e}^{\tau \bar{f}_{d,\varphi,M}(s/\tau,\tau,W)}\, \psi^n(d\tau,dW) &= \int_{[s,+\infty)} \mathrm{e}^{\tau s/\tau \times \left(\frac{\varphi(\tau,W)}{\tau} + d\mathbb{1}_{(M,+\infty]}(\tau)\right)}\, \psi^n(d\tau,dW) \\
&= \int_{[s,+\infty)} \mathrm{e}^{\frac{s}{\tau}\left(\varphi(\tau,W) + d\tau\mathbb{1}_{(M,+\infty]}(\tau)\right)}\, \psi^n(d\tau,dW) \\
&\leq \int_{[s,+\infty)} \mathrm{e}^{1\times\left(\|\varphi\|_\infty + d\tau\mathbb{1}_{(M,+\infty]}(\tau)\right)}\, \psi^n(d\tau,dW) \\
&\leq \mathrm{e}^{\|\varphi\|_\infty} \int_{[s,+\infty)} \mathrm{e}^{d\tau\mathbb{1}_{(M,+\infty]}(\tau)}\, \psi^n(d\tau,dW) \\
&\leq \mathrm{e}^{\|\varphi\|_\infty}\, \psi^n(\mathrm{e}^{d\tau}).
\end{aligned}$$

This integral is bounded uniformly in s, so

$$D_{n,f_{d,\varphi,M}} = \sup_{s>0} \int_{(s,+\infty)} \mathrm{e}^{\tau \bar{f}_{d,\varphi,M}(s/\tau,\tau,W)}\, \psi^n(d\tau,dW) < \infty.$$

Since $\psi^n(e^\varphi) = \gamma < 1$, then there exists M_0 depending on c and φ such that

$$\begin{aligned}
\forall M > M_0, C_{n,f_{d,\varphi,M}} &= \psi^n(\mathrm{e}^{\tau \bar{f}_{d,\varphi,M}(1,\tau,W)}) \\
&= \psi^n(\mathrm{e}^{\varphi(\tau,W) + d\tau\mathbb{1}_{(M,+\infty]}(\tau)}) \\
&= \psi^n(\mathrm{e}^{\varphi(\tau,W)}\, \mathbb{1}_{\tau\leq M} + \mathrm{e}^{d\tau+\varphi(\tau,W)}\, \mathbb{1}_{\tau>M}) \\
&\leq \psi^n(\mathrm{e}^{\varphi}) + \mathrm{e}^{\|\varphi\|_\infty}\, \psi^n(\mathrm{e}^{d\tau}\, \mathbb{1}_{\tau>M}) < 1.
\end{aligned}$$

Therefore, $f_{d,\varphi,M} \in \Gamma$.
Let $\mu \in \Delta^n$, $\mu = \alpha\mu_0 + (1-\alpha)\delta_{(+\infty,+\infty)} \otimes \eta$. Since φ has a compact support:

$$\begin{aligned}
\mu(f_{d,\varphi,M}) &= \alpha\mu_0(f_{d,\varphi,M}) + (1-\alpha)(\delta_{(+\infty,+\infty)} \otimes \eta)(f_{d,\varphi,M}) \\
&= \alpha \int_{[0,1]\times(0,+\infty)\times\{1,\ldots,n\}} f_{d,\varphi,M}(u\tau,(1-u)\tau,W)(da,db,dc)du \otimes \pi(d\tau,dW) + (1-\alpha)d \\
&= \alpha \int_{[0,1]\times(0,+\infty)\times\{1,\ldots,n\}} \frac{\varphi(\tau,W)}{\tau} + d\mathbb{1}_{(M,+\infty]}(\tau)du \otimes \pi(d\tau,dW) + (1-\alpha)d \\
&= \alpha\pi\left(\frac{\varphi(\tau,W)}{\tau}\right) + \alpha d\pi(\tau > M) + (1-\alpha)d \\
&\geq \alpha\pi\left(\frac{\varphi(\tau,W)}{\tau}\right) + (1-\alpha)d.
\end{aligned}$$

We know that $\pi(\varphi/\tau) = \pi(1/\tau)\tilde{\pi}(\varphi)$, then by lemma 2.A.1,

$$\begin{aligned}
\sup_{f\in\Gamma}\mu(f) &\geq \sup_{\gamma<1}\sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})=\gamma}\sup_{d<\xi}\sup_{M>M_0}\mu(f_{d,\varphi,M}) \\
&\geq \sup_{\gamma<1}\sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})=\gamma}\sup_{d}\sup_{M}\left\{\alpha\pi\left(\frac{\varphi(\tau,W)}{\tau}\right)+(1-\alpha)d\right\} \\
&\geq \sup_{\gamma<1}\sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})=\gamma}\sup_{d}\left\{\alpha\pi\left(\frac{\varphi(\tau,W)}{\tau}\right)+(1-\alpha)\right\} \\
&\geq \alpha\sup_{\gamma<1}\sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})=\gamma}\left\{\pi\left(\frac{\varphi(\tau,W)}{\tau}\right)\right\}+(1-\alpha)\xi \\
&\geq \alpha\sup_{\gamma<1}\sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})=\gamma}\left\{\pi\left(\frac{1}{\tau}\right)[\tilde{\pi}(\varphi(\tau,W))-\ln\psi^n(\mathrm{e}^{\varphi})+\ln\gamma]\right\}+(1-\alpha)\xi \\
&\geq \alpha\sup_{\gamma<1}\left(\sup_{\varphi,\psi^n(\mathrm{e}^{\varphi})=\gamma}\left\{\pi\left(\frac{1}{\tau}\right)[\tilde{\pi}(\varphi(\tau,W))-\ln\psi^n(\mathrm{e}^{\varphi})]\right\}+\pi\left(\frac{1}{\tau}\right)\ln\gamma\right)+(1-\alpha)\xi \\
&\geq \alpha\pi\left(\frac{1}{\tau}\right)H(\varphi|\psi^n)+\sup_{\gamma<1}\pi\left(\frac{1}{\tau}\right)\ln\gamma+(1-\alpha)\xi \\
&\geq \alpha\pi\left(\frac{1}{\tau}\right)H(\varphi|\psi^n)+(1-\alpha)\xi = I^n(\mu)
\end{aligned}$$

Then, we have $\forall\mu\in\Delta^n$, $I^n(\mu)\leq\sup_{f\in\Gamma}\mu(f)$. □

Lemma 2.A.6 (Lemma 2.7 from [52]). For all $\mu\in\bar{\Delta}^n\backslash\Delta^n, \sup_{f\in\Gamma}\mu(f)=+\infty$.

Proof. To prove this equality, we determine for each $\mu\in\bar{\Delta}^n\backslash\Delta^n$ an adequate set of functions f_ε, such that f_ε is in Γ and such that $\mu(f_\varepsilon)$ tends to the infinity when ε tends to 0.

Let $\mu\in\bar{\Delta}^n\backslash\Delta^n$: there exist corresponding α, η and π. Since $\mu\notin\Delta^n$, $\alpha>0$ and $\pi(1/\tau)=+\infty$. Fix constants δ, d, D and $M>0$ such that:

$$\psi^n(\tau\in(0,\delta])(\mathrm{e}^d-1)<\psi^n(\tau\in(\delta,M])(1-\mathrm{e}^{-D}).$$

Such constants exist because this inequality is verified when d and $\delta\to0$, and M and $D\to\infty$. For $\varepsilon\in(0,\delta)$, let f_ε be defined by:

$$f_\varepsilon(a,b,c)=\frac{d}{a+b}\mathbb{1}_{(\varepsilon,\delta]}(a+b)-\frac{D}{a+b}\mathbb{1}_{(\delta,M]}(a+b).$$

First, we prove that f_ε is in Γ.

$$\begin{aligned}
\bar{f}_\varepsilon(x,\tau,W) &= \int_0^x f_\varepsilon(u\tau,(1-u)\tau,W)du \\
&= \int_0^x \frac{d}{\tau}\mathbb{1}_{(\varepsilon,\delta]}(\tau)-\frac{D}{\tau}\mathbb{1}_{(\delta,M]}(\tau)du \\
&= x\left(\frac{d}{\tau}\mathbb{1}_{(\varepsilon,\delta]}(\tau)-\frac{D}{\tau}\mathbb{1}_{(\delta,M]}(\tau)\right).
\end{aligned}$$

Then

$$\begin{aligned}
C_{n,f_\varepsilon} &= \int_{(0,+\infty)\times\{1,\dots,n\}} \mathrm{e}^{\tau\bar{f}(1,\tau,W)}\,\psi^n(d\tau,dW)\\
&= \int_{(0,+\infty)\times\{1,\dots,n\}} \mathrm{e}^{d\mathbb{1}_{(\varepsilon,\delta]}(\tau)-D\mathbb{1}_{(\delta,M]}(\tau)}\,\psi^n(d\tau,dW)\\
&= \int_{(0,+\infty)\times\{1,\dots,n\}} \left(\mathrm{e}^0\,\mathbb{1}_{\tau\le\varepsilon}+\mathrm{e}^d\,\mathbb{1}_{(\varepsilon,\delta]}(\tau)+\mathrm{e}^{-D}\,\mathbb{1}_{(\delta,M]}(\tau)+\mathrm{e}^0\,\mathbb{1}_{\tau>M}\right)\psi^n(d\tau,dW)\\
&= 1+\underbrace{(\mathrm{e}^d-1)\psi^n(\tau\in(\varepsilon,\delta])+(\mathrm{e}^{-D}-1)\psi^n(\tau\in(\delta,M])}_{<0}<1.
\end{aligned}$$

and for $s>0$

$$\begin{aligned}
\int_{(s,+\infty)} \mathrm{e}^{\tau\bar{f}_\varepsilon(s/\tau,\tau,W)}\,\psi^n(d\tau,dW) &= \int_{(s,+\infty)} \mathrm{e}^{\tau s/\tau\left(\frac{d}{\tau}\mathbb{1}_{(\varepsilon,\delta]}(\tau)-\frac{D}{\tau}\mathbb{1}_{(\delta,M]}(\tau)\right)}\,\psi^n(d\tau,dW)\\
&= \int_{(s,+\infty)} \mathrm{e}^{\frac{s}{\tau}\left(d\mathbb{1}_{(\varepsilon,\delta]}(\tau)-D\mathbb{1}_{(\delta,M]}(\tau)\right)}\,\psi^n(d\tau,dW)\\
&\le \int_{(s,+\infty)} \mathrm{e}^{\frac{s}{\tau}\left(d\mathbb{1}_{(\varepsilon,\delta]}(\tau)\right)}\,\psi^n(d\tau,dW)\\
&\le \int_{(s,+\infty)} \mathrm{e}^{d}\,\psi^n(d\tau,dW)\\
&\le \mathrm{e}^d,
\end{aligned}$$

so $D_{n,f_\varepsilon}<+\infty$. Then $f_\varepsilon\in\Gamma$.
On the other hand,

$$\begin{aligned}
\mu(f_\varepsilon) &= \alpha\pi\left(\frac{d}{\tau}\mathbb{1}_{(\varepsilon,\delta]}(\tau)-\frac{D}{\tau}\mathbb{1}_{(\delta,M]}(\tau)\right)\\
&= \alpha d\pi\left(\frac{1}{\tau}\mathbb{1}_{(\varepsilon,\delta]}(\tau)\right)-\alpha D\pi\left(\frac{1}{\tau}\mathbb{1}_{(\delta,M]}(\tau)\right).
\end{aligned}$$

Since $\pi(1/\tau)=+\infty$, $\pi\left(\frac{1}{\tau}\mathbb{1}_{\tau\in(0,\delta]}\right)=+\infty$ and $\mu(f_\varepsilon)\underset{\varepsilon\to0}{\longrightarrow}+\infty$.
Then $\forall\mu\in\bar{\Delta}^n\backslash\Delta^n, \sup_{f\in\Gamma}\mu(f)=+\infty$. □

Proposition 2.A.7 (Equivalent of the proposition 3.3). For all f in $\Gamma=\{f:(0,+\infty]^2\times\{1,...,n\}\to\mathbb{R},$ bounded and lower continuous such that $C_{n,f}<1, D_{n,f}<+\infty\}$,

$$\sup_{t>0}\mathbb{E}(\mathrm{e}^{t\mu_t^n(f)})<+\infty \tag{2.A.6}$$

Proof. Since $C_{n,f}\in(0,+\infty)$, we can introduce the probability measure $\psi_f^n(d\tau,dV)=1/C_{n,f}\times\mathrm{e}^{\tau\bar{f}(1,\tau,V)}\,\psi^n(d\tau,dV)$. We denote by ζ_k the law of S_k if $(\tau_i)_i$ is i.i.d with common law ψ_f^n.

$$\begin{aligned}
&\mathbb{E}(\mathrm{e}^{t\mu_t^n(f)}) \\
&\quad = \mathbb{E}\left[\exp\left(\int_0^t f(A_s, B_s, C_s^n)ds\right)\right] \\
&\quad = \mathbb{E}\left[\mathbb{1}_{M_t=0}\exp\left(\tau_1 \bar{f}\left(\frac{t}{\tau_1}, \tau_1, W_1^n\right)\right)\right] \\
&\qquad + \sum_{k=1}^{\infty}\mathbb{E}\left[\mathbb{1}_{M_t=k}\exp\left(\sum_{i=1}^{k}\tau_i\bar{f}(1,\tau_i,W_i^n) + \tau_{k+1}\bar{f}\left(\frac{t-S_k}{\tau_{k+1}}, \tau_{k+1}, W_{k+1}^n\right)\right)\right] \\
&\quad = \int_{(t,+\infty)\times\{1,\dots,n\}} \mathrm{e}^{\tau\bar{f}(1/\tau,\tau,V)}\,\psi^n(d\tau,dV) \\
&\qquad + \sum_{k=1}^{+\infty}\mathbb{E}\left[\mathbb{1}_{M_t=k}\prod_{i=1}^{k}\mathrm{e}^{\tau_i\bar{f}(1,\tau_i,W_i^n)}\exp\left(\tau_{k+1}\bar{f}\left(\frac{t-S_k}{\tau_{k+1}}, \tau_{k+1}, W_{k+1}^n\right)\right)\right] \\
&\quad = \int_{(t,+\infty)\times\{1,\dots,n\}} \mathrm{e}^{\tau\bar{f}(1/\tau,\tau,V)}\,\psi^n(d\tau,dV) \\
&\qquad + \sum_{k=1}^{+\infty}\int_{[0,t]}\int_{(t-s,+\infty)} C_{n,f}^k\,\mathrm{e}^{\tau\bar{f}((t_s/\tau,\tau,V))}\,\psi^n(d\tau,dV)\zeta_k(ds) \\
&\quad \leq D_f + \sum_{k=1}^{+\infty}C_{n,f}^k D_{n,f} = \frac{D_{n,f}}{1-C_{n,f}}.
\end{aligned}$$

Then $\sup_{t>0}\mathbb{E}(\mathrm{e}^{t\mu_t^n(f)}) < +\infty$. □

2.A.3 Complement to Section 2.7 *Proof of Theorem 2.4.3*

2.A.3.1 Upper bound

In the proof of the upper bound, we use the Lemma 2.7.1, which is a part of Lemma 3.1 from [52]:

Lemma 2.A.8 (Lemma 2.7.1)*.* We have:

$$\lim_{M\to+\infty}\limsup_{t\to+\infty}\frac{1}{t}\ln\mathbb{P}\left(\mu_t^n\left(\frac{1}{a+b}\right) > M\right) = -\infty.$$

Proof. Let $M > 0$. We recall that $\{S_n \leq t\} = \{M_t + 1 > n\}$ and $\frac{t-S_{M_t}}{\tau_{M_t+1}} \leq 1$. If $\lfloor Mt \rfloor \geq 1$, we have:

$$\begin{aligned}
\left\{\mu_t^n\left(\frac{1}{a+b}\right) > M\right\} &= \left\{M_t + \frac{t-S_{M_t}}{\tau_{M_t+1}} > Mt\right\} \\
&\subset \{M_t + 1 > \lfloor Mt \rfloor\} \\
&\subset \left\{S_{\lfloor Mt \rfloor} \leq t\right\}.
\end{aligned}$$

Therefore, by the Markov inequality:

$$\mathbb{P}\left(\mu_t^n\left(\frac{1}{a+b}\right) > M\right) \leq \mathbb{P}\left(S_{\lfloor Mt \rfloor} \leq t\right)$$
$$\leq \mathbb{E}\left[\mathrm{e}^{-S_{\lfloor Mt \rfloor}}\right]\mathrm{e}^t = \mathbb{E}\left[\mathrm{e}^{-\tau_1}\right]^{\lfloor Mt \rfloor}\mathrm{e}^t .$$

Then, we have:

$$\begin{aligned}\frac{1}{t}\ln\mathbb{P}\left(\mu_t^n\left(\frac{1}{a+b}\right) > M\right) &\leq \frac{1}{t}\left(t + \lfloor Mt \rfloor \ln\mathbb{E}\left[\mathrm{e}^{-\tau_1}\right]\right)\\ &= 1 + \frac{\lfloor Mt \rfloor}{t}\ln\mathbb{E}\left[\mathrm{e}^{-\tau_1}\right]\\ &\xrightarrow[t\to\infty]{} 1 + M\ln\mathbb{E}\left[\mathrm{e}^{-\tau_1}\right].\end{aligned}$$

Eventually, we have:

$$\lim_{M\to+\infty}\overline{\lim_{t\to+\infty}}\frac{1}{t}\ln\mathbb{P}\left(\mu_t^n\left(\frac{1}{a+b}\right) > M\right) = -\infty.$$

□

We also prove that the empirical measure μ_t^n is close to Δ^n, and we give the Lemma 2.7.3

Lemma 2.A.9 (Lemma 2.7.3 - Analogue of Lemma 3.2 in [52] - Slightly different proof). For $f \in C_b((0,+\infty]^2 \times \{1,...,n\})$ we define

$$\nu_t^n(f) := \frac{1}{t}\sum_{i=1}^{M_t}\tau_i\int_0^1 f(u\tau_i,(1-u)\tau_i,W_i^n)du + \frac{t-S_{M_t}}{t}f(+\infty,+\infty,W_{M_t+1}^n).$$

Then $\nu_t^n \in \Delta^n$ almost surely.

For all g, δ there exists some $t(g,\delta)$ such that for $t \geq t(g,\delta)$, the events $\{|\mu_t^n(g) - \nu_t^n(g)| > \delta\}$ and $\left\{\left|\mu_t^n\left(\frac{1}{a+b}\right) - \nu_t^n\left(\frac{1}{a+b}\right)\right| > \delta\right\}$ are almost surely empty.

Proof. We can see that $\nu_t^n \in \Delta^n$ a.s. with $\alpha = \frac{S_{M_t}}{t} \in [0,1]$, $\pi^n = \frac{1}{S_{M_t}}\sum_{i=1}^{M_t}\tau_i\delta_{(\tau_i,W_i^n)}$ and $\eta = \delta_{W_{M_t+1}^n}$. If $S_{M_t} = 0$ (ie $M_t = 0$), then $\alpha = 0$, so π^n can be any probability measure such that $\pi^n(1/\tau) < \infty$.
Else,

$$\begin{aligned}\pi^n\left(\frac{1}{\tau}\right) &= \frac{1}{S_{M_t}}\sum_{i=1}^{M_t}\tau_i\frac{1}{\tau_i}\\ &= \frac{M_t}{S_{M_t}} < \infty \text{ a.s.}\end{aligned}$$

Now, let $f \in \mathcal{C}_b((0,+\infty]^2 \times \{1,...,n\})$. We remind the decomposition (2.A.5):

$$\mu_t^n(f) = \frac{1}{t}\sum_{i=1}^{M_t} \tau_i \bar{f}(1,\tau_i,W_i^n) + \frac{\tau_{M_t+1}}{t}\bar{f}\left(\frac{t-S_{M_t}}{\tau_{M_t+1}}, \tau_{M_t+1}, W_{M_t+1}^n\right)$$

with $\bar{f}(x,\tau,w) = \int_0^x f(u\tau,(1-u)\tau,w)du$.

We can rewrite the second term

$$\frac{\tau_{M_t+1}}{t}\bar{f}\left(\frac{t-S_{M_t}}{\tau_{M_t+1}}, \tau_{M_t+1}, W_{M_t+1}^n\right) = \frac{t-S_{M_t}}{t}\int_0^1 f(u(t-S_M), \tau_{M_t+1}-u(t-S_M), W_{M_t+1}^n)du.$$

Then:

$$\begin{aligned}&|\mu_t^n(f) - \nu_t^n(f)| \\ &= \frac{t-S_{M_t}}{t}\left|\int_0^1 f(u(t-S_{M_t}), \tau_{M_t+1}-u(t-S_{M_t}), W_{M_t+1}^n) - f(+\infty,+\infty,W_{M_t+1}^n)du\right|\end{aligned}$$

We have $t - S_{M_t} \leq \tau_{M_t+1}$ by definition.
We denote

$$\begin{aligned}\xi(s,k) &:= \int_0^1 \sup_{T\geq s}|f(us,T-us,k) - f(+\infty,+\infty,k)|du, \\ E(s) &:= \sup_{u\geq s, k\leq n} \xi(u,k).\end{aligned}$$

We have $f \in \mathcal{C}_b((0,+\infty]^2 \times \{1,...,n\})$ so $f(a,b,c) \to f(+\infty,+\infty,c)$ as $(a,b) \to (+\infty,+\infty)$. f is bounded so ξ is well defined and finite. Moreover, ξ is bounded (and E too).
For each k, $\xi(s,k)$ tends to 0 as $s \to +\infty$. So, for $u \in (0,1)$,

$$f(us,T-us,k) \underset{s\to\infty, T\geq s}{\longrightarrow} f(+\infty,+\infty,k),$$

and

$$\sup_{T\geq s}|f(us,T-us,k) - f(+\infty,+\infty,k)| \underset{s\to\infty}{\longrightarrow} 0.$$

The Fatou's lemma allows us to conclude:

$$\begin{aligned}&\lim_{s\to+\infty}\int_0^1 \sup_{T\geq s}|f(us,T-us,k) - f(+\infty,+\infty,k)|du \\ &\quad\leq \int_0^1 \liminf_{s\to+\infty}\sup_{\tau\geq s}|f(us,\tau-us,k) - f(+\infty,+\infty,k)|du = 0\end{aligned}$$

For each $k \leq n$, $\xi(.,k)$ is bounded by a constant M and tends to 0 as $s \to \infty$, so E is bounded by M, tends to 0 as $s \to \infty$ and is monotone non-increasing.

Then:

$$\begin{aligned}
&|\mu_t^n(f) - \nu_t^n(f)| \\
&\leq \frac{t - S_{M_t}}{t} \int_0^1 \left| f(u(t - S_{M_t}), \tau_{M_t+1} - u(t - S_{M_t}), W^n_{M_t+1}) - f(+\infty, +\infty, W^n_{M_t+1}) \right| du \\
&\leq \frac{t - S_{M_t}}{t} \xi(t - S_{M_t}, W^n_{M_t+1}) \\
&\leq \frac{t - S_{M_t}}{t} E(t - S_{M_t}).
\end{aligned}$$

Finally

$$\begin{aligned}
\{|\mu_t^n(f) - \nu_t(f)| > \delta\} \subset & \left\{ \frac{t - S_{M_t}}{t} E(t - S_{M_t}) > \delta \right\} \\
&= \{x_t E(x_t \times t) > \delta\},
\end{aligned}$$

where $x_t = \frac{t - S_{M_t}}{t} \in [0,1]$.
If $x \in [0,1]$ satisfies $xE(tx) > \delta$, then $E(tx) > \delta$. Then $x \leq E^{-1}(\delta)/t$. So that $\delta < xE(tx) < E^{-1}(\delta)/t \times M$. When $t \geq ME^{-1}(\delta)/\delta$, this is impossible. Therefore, for t large enough, the event $\{|\mu_t^n(f) - \nu_t^n(f)| > \delta\}$ is empty.

The proof for the function $(a,b,c) \mapsto \frac{1}{a+b}$ is already given in the proof following Lemma 2.7.3. □

Structure of the Proof of the upper bound

We give all the ingredients of the proof in the Section 2.7, and we refer to [52] for the proof. Here, the structure of this proof:

<u>Study of $\mathbb{P}_t^n(\mathcal{O})$ for an open set $\mathcal{O}$ to exhibit a first rate function:</u>
For $\mathcal{A}$ measurable subset of $\mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$ and $f \in \Gamma$, we have:

$$\begin{aligned}
\frac{1}{t} \ln \mathbb{P}_t^n(\mathcal{A}) &= \frac{1}{t} \ln \mathbb{E}\left[\mathrm{e}^{t\mu_t^n(f)} \, \mathrm{e}^{-t\mu_t^n(f)} \, \mathbb{1}_{\mathcal{A}}(\mu_t^n) \right] \\
&\leq \frac{1}{t} \ln \mathbb{E}\left[\mathrm{e}^{t\mu_t^n(f)} \, \mathrm{e}^{-t \inf_{\mu \in \mathcal{A}} \mu(f)} \right] \\
&\leq \frac{1}{t} \ln \mathbb{E}\left[\mathrm{e}^{t\mu_t^n(f)} \right] - \inf_{\mu \in \mathcal{A}} \mu(f) \\
&\leq \frac{1}{t} \ln \frac{D_{n,f}}{1 - C_{n,f}} - \inf_{\mu \in \mathcal{A}} \mu(f),
\end{aligned}$$

where the last inequality is obtained thanks to the Lemma 2.6.5. Thus we have:

$$\limsup_{t \to \infty} \frac{1}{t} \ln \mathbb{P}_t^n(\mathcal{A}) \leq - \inf_{\mu \in \mathcal{A}} \mu(f).$$

Let $\mathcal{O}$ be an open subset of $\mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$. We apply this last inequality for $\mathcal{A} = \mathcal{O} \cap \Delta^n_{M,g,\delta}$:

$$\begin{aligned}\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}^n_t(\mathcal{O}) &\leq \limsup_{t\to+\infty} \frac{1}{t} \ln \left(2 \max(\mathbb{P}^n_t(\mathcal{O} \cap \Delta^n_{M,g,\delta}), \mathbb{P}^n_t((\Delta^n_{M,g,\delta})^c))\right) \\ &\leq \max\left(\limsup_{t\to+\infty} \frac{1}{t} \ln(\mathbb{P}^n_t(\mathcal{O} \cap \Delta^n_{M,g,\delta})), -R^n_{M,g,\delta}\right) \\ &\leq -\left(\inf_{\mu\in\mathcal{O}\cap\Delta^n_{M,g,\delta}} \mu(f) \wedge R^n_{M,g,\delta}\right).\end{aligned}$$

Recall the definition of $I^n_{f,M,g,\delta}(\mu)$ in (2.7.3), for all g, M, δ and all $f \in \Gamma$,

$$I^n_{f,M,g,\delta}(\mu) = \begin{cases} \mu(f) \wedge R^n_{M,g,\delta} & \text{if } \mu \in \Delta^n_{M,g,\delta} \\ +\infty & \text{otherwise.} \end{cases},$$

thus, as announced in Section 2.7 , for all open set $\mathcal{O}$, all g, M, δ and all $f \in \Gamma$,

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}^n_t(\mathcal{O}) \;\leq\; -\inf_{\nu\in\mathcal{O}} I^n_{f,M,g,\delta}(\mu). \tag{2.A.7}$$

Large deviations upper bound on compact sets

By Lemma 2.B.2, since f is lower semicontinuous, $I^n_{f,M,g,\delta}(\mu)$ is also lower semicontinuous thanks to the compactness of $\Delta^n_{M,g,\delta}$ (Lemma 2.7.2).

By minimizing (2.A.7) over $\{f, \delta, g, M\}$ we obtain:

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}^n_t(\mathcal{O}) \leq -\sup_{f,M,g,\delta} \inf_{\mu\in\mathcal{O}} I^n_{f,M,g,\delta}(\mu).$$

We apply the minimax lemma with $-I^n_{f,M,g,\delta}$ [Kipnis Landim, Appendix 2.3, Lemma 2.3, [51]] on this equation and we get for all compact set $\mathcal{K}$

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}^n_t(\mathcal{O}) \leq -\inf_{\mu\in\mathcal{K}} \sup_{f,M,g,\delta} I^n_{f,M,g,\delta}(\mu).$$

So $(\mathbb{P}^n_t)_{t\geq 0}$ satisfies a large deviations upper bound on compact sets with speed t and rate $\tilde{I}(\mu)$ for $\mu \in \mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$ where

$$\tilde{I}(\mu) := \sup\{I^n_{f,M,g,\delta}(\mu), f \in \Gamma, M > 0, g \in \mathcal{C}_b((0,+\infty]^2 \times \{1,...,n\}), \delta > 0\}.$$

Conclusion: As proved in Section 2.7, we have

$$\tilde{I}(\mu) \geq \sup_{f\in\Gamma} I_f(\mu),$$

where

$$I_f(\mu) = \begin{cases} \mu(f) & \text{if } \mu \in \bar{\Delta}^n \\ +\infty & \qquad\qquad \text{otherwise.} \end{cases}.$$

According to lemma 2.6.4, $\tilde{I}(\mu) \geq I^n(\mu)$.
Then $(\mathbb{P}^n_t)_{t\geq 0}$ satisfies a large deviations upper bound with rate I^n on compact sets. The exponential tightness of $(\mu_t)_t$ follows the Lemma 2.7.1. By the Lemma 1.2.18 from [Dembo Zeitouni, [25]], $(\mathbb{P}^n_t)_{t\geq 0}$ satisfies the full large deviations upper bound on closed sets.

2.A.3.2 Lower bound

The proof of the lower bound is very similar to [52], just replacing the sample τ_i by a sample (τ_i, W_i).

We prove:

Theorem 2.A.10 (Lower bound). For each open set $\mathcal{O} \subset \mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$,

$$\liminf \frac{1}{t} \ln \mathbb{P}^n_t(\mathcal{O}) \geq -\inf_{\mu\in\mathcal{O}} I^n(\mu).$$

The proof of this bound is much more simple than the upper bound. We only need the following Proposition:

Proposition 2.A.11 (Equivalent of Proposition 4.2). For every $\mu \in \Delta^n$, there exists a family $\mathbb{Q}_t$ of probability measures on $\mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$ such that $\mathbb{Q}_t \rightharpoonup \delta_\mu$ and

$$\limsup_{t\to+\infty} \frac{1}{t} H(\mathbb{Q}_t|\mathbb{P}_t) \leq I^n(\mu).$$

The proof of this Proposition will be given at the end of this part. We can now give the proof of the lower bound:

Proof of the lower bound. Let $\mu \in \Delta^n$ and $\mathcal{V}$ be an open neighborhood of μ in the weak topology. By the proposition 2.A.11, there exists $\mathbb{Q}_t$ such that $\mathbb{Q}_t \rightharpoonup \delta_\mu$ and:

$$\limsup_{t\to+\infty} \frac{1}{t} H(\mathbb{Q}_t|\mathbb{P}^n_t) \leq I^n(\mu).$$

Then, by using Jensen's inequality:

$$\begin{aligned}
\ln \mathbb{P}_t^n(\mathcal{V}) &= \ln\left(\int_{\mathcal{V}} \frac{d\mathbb{P}_t^n}{d\mathbb{Q}_t} d\mathbb{Q}_t\right) \\
&= \ln\left(\frac{1}{\mathbb{Q}_t(\mathcal{V})}\int_{\mathcal{V}} \frac{d\mathbb{P}_t^n}{d\mathbb{Q}_t} d\mathbb{Q}_t\right) + \ln \mathbb{Q}_t(\mathcal{V}) \\
&\geq \frac{1}{\mathbb{Q}_t(\mathcal{V})}\int_{\mathcal{V}} \ln\left(\frac{d\mathbb{P}_t^n}{d\mathbb{Q}_t}\right) d\mathbb{Q}_t + \ln \mathbb{Q}_t(\mathcal{V}) \\
&\geq -\frac{1}{\mathbb{Q}_t(\mathcal{V})}\int_{\mathcal{V}} \ln\left(\frac{d\mathbb{Q}_t}{d\mathbb{P}_t^n}\right)\frac{d\mathbb{Q}_t}{d\mathbb{P}_t^n} d\mathbb{P}_t^n + \ln \mathbb{Q}_t(\mathcal{V}) \\
&\geq \frac{1}{\mathbb{Q}_t(\mathcal{V})}\left(-H(\mathbb{Q}_t|\mathbb{P}_t^n) + \int_{\mathcal{V}^c} \ln\left(\frac{d\mathbb{Q}_t}{d\mathbb{P}_t^n}\right)\frac{d\mathbb{Q}_t}{d\mathbb{P}_t^n} d\mathbb{P}_t^n\right) + \ln \mathbb{Q}_t(\mathcal{V}).
\end{aligned}$$

Moreover, $\forall x \geq 0, x\ln x \geq -\mathrm{e}^{-1}$ and we obtain:

$$\begin{aligned}
\ln \mathbb{P}_t^n(\mathcal{V}) &\geq \frac{1}{\mathbb{Q}_t(\mathcal{V})}\left(-H(\mathbb{Q}_t|\mathbb{P}_t^n) - \int_{\mathcal{V}^c} \mathrm{e}^{-1}\, d\mathbb{P}_t^n\right) + \ln \mathbb{Q}_t(\mathcal{V}) \\
&\geq \frac{1}{\mathbb{Q}_t(\mathcal{V})}\left(-H(\mathbb{Q}_t|\mathbb{P}_t^n) - \mathrm{e}^{-1}\right) + \ln \mathbb{Q}_t(\mathcal{V}).
\end{aligned}$$

However, $\mathbb{Q}_t \rightharpoonup \delta_\mu$, $\mu \in \mathcal{V}$ which is open, then $\mathbb{Q}_t(\mathcal{V}) \underset{t\to\infty}{\longrightarrow} 1$. We obtain:

$$\liminf_{t\to+\infty} \frac{1}{t}\ln \mathbb{P}_t^n(\mathcal{V}) \geq -\limsup_{t\to\infty} \frac{1}{t} H(\mathbb{Q}_t|\mathbb{P}_t^n) \geq -I^n(\mu).$$

□

Proof of Proposition 2.A.11

To prove this Proposition, we just need a law of large numbers for μ_t^n. We denote by $\mathbb{P}_{\tilde{\pi}}$ the law of an i.i.d. sequence of $(\tau_i, W_i)_i$ with marginal distribution $\tilde{\pi}$.

Proposition 2.A.12 (Proposition 4.1 of the article). Let $\pi \in \mathcal{M}^1((0,+\infty]\times\{1,...,n\})$ with $\pi(1/\tau) \in (0,+\infty)$. Under $\mathbb{P}_{\tilde{\pi}}$, a.s,

$$\mu_t^n \underset{t\to\infty}{\rightharpoonup} \int_{[0,1]\times(0,+\infty)\times\{1,...,n\}} \delta_{(u\tau,(1-u)\tau,V)} du \otimes \pi(d\tau, dV).$$

Proof. With the law of the large numbers and the renewal theorem, we show that under $\mathbb{P}_{\tilde{\pi}}$, for each $f \in \mathcal{C}_b((0,+\infty]^2\times\{1,...,n\})$, $\mu_t^n(f)$ converges a.s.

Let $f \in \mathcal{C}_b((0,+\infty]^2\times\{1,...,n\})$, we remind (2.A.5),

$$\mu_t^n(f) = \frac{1}{t}\sum_{i=1}^{M_t} \tau_i \bar{f}(1,\tau_i,W_i^n) + \frac{\tau_{M_t+1}}{t}\bar{f}\left(\frac{t-S_{M_t}}{\tau_{M_t+1}}, \tau_{M_t+1}, W_{M_t+1}^n\right).$$

By the strong law of large numbers, under $\mathbb{P}_{\tilde{\pi}}$, a.s.

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} \tau_i \bar{f}(1,\tau_i,W_i^n) = \tilde{\pi}(\tau\bar{f}(1,\tau,W^n)) = \frac{1}{\pi(1/\tau)}\pi(\bar{f}(1,\tau,W^n)).$$

By the renewal theorem, a.s.

$$\lim_{t\to+\infty}\frac{M_t}{t}=\frac{1}{\mathbb{E}_{\tilde\pi}(\tau)}=\frac{\pi(1/\tau)}{\int\tau\frac{1}{\tau}\pi(d\tau)}=\pi(1/\tau)$$

Therefore, under $\mathbb{P}_{\tilde\pi}$ a.s.

$$\begin{aligned}\lim_{t\to+\infty}\frac{1}{t}\sum_{i=1}^{M_t}\tau_i\bar f(1,\tau_i,W_i^n)&=\lim_{t\to+\infty}\frac{M_t}{t}\frac{1}{M_t}\sum_{i=1}^{M_t}\tau_i\bar f(1,\tau_i,W_i^n)\\&=\pi(\bar f(1,\tau,W^n))\end{aligned}$$

By the law of large numbers, a.s.

$$\lim_{k\to+\infty}\frac{S_k}{k}=\tilde\pi(\tau)=\frac{1}{\pi(1/\tau)},$$

so, a.s,

$$\lim_{t\to+\infty}\frac{S_{M_t}}{t}=\lim_{t\to+\infty}\frac{S_{M_t}}{M_t}\frac{M_t}{t}=1 \text{ and } \lim_{t\to+\infty}\frac{t-S_{M_t}}{t}=0.$$

It follows that a.s.

$$\lim_{t\to+\infty}\left|\frac{\tau_{M_t+1}}{t}\bar f\left(\frac{t-S_{M_t}}{\tau_{M_t+1}},\tau_{M_t+1},W^n_{M_t+1}\right)\right|\le\lim_{t\to+\infty}\frac{t-S_{M_t}}{t}\|f\|_\infty=0.$$

So for all $f\in\mathcal{C}_b((0,+\infty]^2\times\{1,...,n\})$, $\mathbb{P}_{\tilde\pi}$ a.s.

$$\mu_t^n(f)\underset{t\to\infty}{\longrightarrow}\pi(\bar f(1,\tau,W^n))=\int_{[0,1]\times(0,+\infty)\times\{1,...,n\}}f(u\tau,(1-u)\tau,V)du\otimes\pi(d\tau,dV).$$

□

Proof of Proposition 2.A.11. First, we prove this proposition for $\mu\in\Delta_0^n$. We construct a sequence $\mathbb{Q}_t^{n,\delta}$ of laws, such that $\lim_{\delta\to0}\lim_{t\to+\infty}\mathbb{Q}_t^{n,\delta}=\delta_\mu$. By using the contraction principle on the relative entropy and selecting a good $\delta(t)$, we can prove that the expression above is true. Eventually, we extend this proposition to Δ^n, by using the proposition 2.5.6: for each $\mu\in\Delta^n$, there exists a good sequence $\mu_k\in\Delta_0^n$.

We suppose $\mu\in\Delta_0^n$, with associated π. Notice that $\mu(1/\tau)=\pi(1/\tau)\in(0,+\infty)$. Fix $\delta>0$ and set $T_{t,\delta}=\lfloor(1+\delta)t\pi(1/\tau)\rfloor$.
For $t>\dfrac{1}{\pi(1/\tau)}$, let us denote by $\mathbb{P}_t^{n,\delta}$ the law on $((0,+\infty)\times\{1,...,n\})^{\mathbb{N}^*}$ such that under $\mathbb{P}_t^{n,\delta}$, the sequence $(\tau_i,W_i)_i$ is independent and:

1. for all $i\le T_{t,\delta}$, (τ_i,W_i) has law $\tilde\pi$,
2. for all $i\ge T_{t,\delta}+1$, (τ_i,W_i) has law ψ.

We set $\mathbb{Q}_t^{n,\delta} = \mathbb{P}_t^{n,\delta} \circ (\mu_t^n)^{-1}$. We will prove that: $\lim_{\delta\to 0}\lim_{t\to+\infty} \mathbb{Q}_t^{n,\delta} = \delta_\mu$.
By the law of large numbers of proposition 2.A.12, under $\mathbb{P}_{\tilde{\pi}}$ we have a.s.

$$\lim_{k\to+\infty} \frac{S_k}{k} = \mathbb{E}_{\tilde{\pi}}(\tau) = \int \tau \frac{1}{\tau\pi(1/\tau)}\pi(d\tau) = \frac{1}{\pi(1/\tau)}.$$

So

$$\lim_{t\to+\infty} \frac{S_{T_{t,\delta}}}{t} = \lim_{t\to+\infty} \frac{S_{T_{t,\delta}}}{T_{t,\delta}} \frac{T_{t,\delta}}{t} = \frac{1}{\pi(1/\tau)}(1+\delta)\pi(1/\tau) = 1+\delta$$

Moreover, $S_{T_{t,\delta}}$ has the same law under $\mathbb{P}_{\tilde{\pi}}$ and $\mathbb{P}_t^{n,\delta}$ so, for any $\delta > 0$,

$$\lim_{t\to+\infty} \mathbb{P}_t^{n,\delta}(S_{T_{t,\delta}} \leq t) = \lim_{t\to+\infty} \mathbb{P}_{\tilde{\pi}}(S_{T_{t,\delta}} \leq t) = \lim_{t\to+\infty} \mathbb{P}_{\tilde{\pi}}\left(\frac{S_{T_{t,\delta}}}{t} \leq 1\right) = 0.$$

We set $D_{t,\delta} = \{S_{T_{t,\delta}} > t\}$. We have for all $\delta > 0$, $\lim_{t\to+\infty} \mathbb{P}_t^{n,\delta}(D_{t,\delta}) = 1$.
We recall that $\{S_n > t\} = \{M_t + 1 \leq n\}$. Therefore, on $D_{t,\delta}$ we have $M_t + 1 \leq T_{t,\delta}$ and therefore for any $f \in \mathcal{C}_b((0,+\infty]^2 \times \{1,...,n\})$ and $\varepsilon > 0$,

$$\mathbb{P}_t^{n,\delta}(|\mu_t^n(f) - \mu(f)| > \varepsilon) \leq \mathbb{P}_t^{n,\delta}(\{|\mu_t^n(f) - \mu(f)| > \varepsilon\} \cap D_{t,\delta}) + \mathbb{P}_t^{n,\delta}(D_{t,\delta}^c)$$

We have $\lim_{t\to+\infty} \mathbb{P}_t^{n,\delta}(D_{t,\delta}^c) = 0$.
On $D_{t,\delta}$, $\mathbb{P}_t^{n,\delta}$ and $\mathbb{P}_{\tilde{\pi}}$ coincide, so

$$\begin{aligned}\mathbb{P}_t^{n,\delta}(\{|\mu_t^n(f) - \mu(f)| > \varepsilon\} \cap D_{t,\delta}) &= \mathbb{P}_{\tilde{\pi}}(\{|\mu_t^n(f) - \mu(f)| > \varepsilon\} \cap D_{t,\delta}) \\ &\leq \mathbb{P}_{\tilde{\pi}}(|\mu_t^n(f) - \mu(f)| > \varepsilon) \longrightarrow_{t\to+\infty} 0,\end{aligned}$$

where we use Proposition 2.A.12.

It implies that, for all $\delta > 0$, $\lim_{t\to+\infty} \mathbb{Q}_t^{n,\delta} = \delta_\mu$. In particular, we have:

$$\lim_{\delta\to 0}\lim_{t\to+\infty} \mathbb{Q}_t^{n,\delta} = \delta_\mu.$$

Now we estimate the entropy $H(\mathbb{Q}_t^{n,\delta}|\mathbb{P}_t)$, by using the contraction principle $H(\mathbb{P}_t^{n,\delta} \circ (\mu_t^n)^{-1} | \mathbb{P}_\psi \circ (\mu_t^n)^{-1}) \leq H(\mathbb{P}_t^{n,\delta}|\mathbb{P}_\psi)$.

$$\begin{aligned}H(\mathbb{Q}_t^{n,\delta}|\mathbb{P}_t) &\leq H(\mathbb{P}_t^{n,\delta}|\mathbb{P}_\psi) \\ &= H\left[\left(\bigotimes_{i\leq T_{t,\delta}} \tilde{\pi}\right) \otimes \left(\bigotimes_{i>T_{t,\delta}} \psi\right) \middle| \left(\bigotimes_{i\leq T_{t,\delta}} \psi\right) \otimes \left(\bigotimes_{i>T_{t,\delta}} \psi\right)\right] \\ &= T_{t,\delta} H(\tilde{\pi}|\psi).\end{aligned}$$

So:

$$\lim_{\delta\to 0}\limsup_{t\to+\infty}\frac{1}{t}H(\mathbb{Q}_t^{n,\delta}|\mathbb{P}_t) \le \lim_{\delta\to 0}\limsup_{t\to+\infty}\frac{1}{t}T_{t,\delta}H(\tilde{\pi}|\psi)$$
$$= \pi(1/\tau)H(\tilde{\pi}|\psi).$$

Then, there exists a function $t \mapsto \delta(t) > 0$ vanishing as $t \to +\infty$ such that $\mathbb{Q}_t = \mathbb{Q}_t^{n,\delta(t)} \to \delta_\mu$ and $\limsup_{t\to+\infty}\frac{1}{t}H(\mathbb{Q}_t|\mathbb{P}_t) \le I^n(\mu)$.

We suppose now $\mu \in \Delta^n \backslash \Delta_0^n$.
By proposition 2.5.6 we can find a sequence $(\mu_k)_k$ in Δ_0^n such that $\mu_k \rightharpoonup \mu$ and $\limsup I^n(\mu_k) \le I^n(\mu)$. Moreover, for all k, there exists a family $\mathbb{Q}_{t,k}$ on $\mathcal{M}^1((0,+\infty]^2\times\{1,...,n\})$ such that $\mathbb{Q}_{t,k} \rightharpoonup \delta_{\mu_k}$ and:

$$\limsup_{t\to+\infty}\frac{1}{t}H(\mathbb{Q}_{t,k}|\mathbb{P}_t) \le I^n(\mu_k)$$

With a diagonal procedure, we can find a family $\mathbb{Q}_t$ such that $\mathbb{Q}_t \rightharpoonup \delta_\mu$ and

$$\limsup_{t\to\infty}\frac{1}{t}H(\mathbb{Q}_t|\mathbb{P}_t) \le I^n(\mu).$$

□

2.A.4 Complement to Section 2.8 *Proof of Theorem 2.4.4*

We recall the notations from Section 2.8.
We define for $\varepsilon > 0$

$$\varphi^\varepsilon(a,b,c) = \frac{c}{(a+b)\vee\varepsilon}.$$

Replacing τ_i by $\tau_i^\varepsilon = \tau_i \vee \varepsilon$ we may introduce S_n^ε, M_t^ε, A_t^ε, B_t^ε, $C_t^{n,\varepsilon}$ and $\mu_t^{n,\varepsilon}$ as in subsection 2.4.1. We have the following

$$\left|\mu_t^{\varepsilon,n}(\varphi^\varepsilon) - \frac{1}{t}\sum_{i=1}^{M_t} W_i^n\right| \quad \le \quad \frac{n}{t}\left((M_t - M_t^\varepsilon) + 1\right).$$

Lemma 2.A.13 (Lemma 5.4 from [52]). $(M_t^\varepsilon/t)_{t\ge 0}$ is an exponentially good approximation of $(M_t)_t$, i.e. for all $\delta > 0$,

$$\lim_{\varepsilon\to 0}\limsup_{t\to+\infty}\frac{1}{t}\ln\mathbb{P}\left(|M_t - M_t^\varepsilon| > t\delta\right) = -\infty$$

Proof. Since $M_t^\varepsilon \le M_t$, we study for $\delta > 0$ and $C > 0$

$$\mathbb{P}\left(M_t - M_t^\varepsilon > t\delta\right) \le \sum_{i=0}^{\lfloor Ct\rfloor}\mathbb{P}\left(M_t - M_t^\varepsilon > t\delta, M_t = i\right) + \mathbb{P}(M_t > Ct)$$
$$= \sum_{i=0}^{\lfloor Ct\rfloor}\sum_{k=0}^{i}\mathbb{P}\left(M_t - M_t^\varepsilon > t\delta, M_t = i, \sum_{j=1}^{i}\mathbb{1}_{\tau_j<\varepsilon} = k\right) + \mathbb{P}(S_{\lfloor Ct\rfloor} \le t).$$

For $k \leq i \leq Ct$ and $m \leq n$, on the event $\{M_t = n, M_t^\varepsilon = m, \sum_{j=1}^{i} \mathbb{1}_{\tau_j<\varepsilon} = k\}$, we have:
$t < S_m^\varepsilon \leq S_m + k\varepsilon$, then $S_m > t - k\varepsilon$, and finally $M_{t-k\varepsilon} \leq m = M_t^\varepsilon$. Eventually, since $k \leq Ct$, $M_{t-Ct\varepsilon} \leq M_t^\varepsilon$. Therefore

$$\mathbb{P}\left(M_t - M_t^\varepsilon > t\delta\right) \leq \frac{(Ct^2)}{2}\mathbb{P}\left(M_t - M_{t-Ct\varepsilon} > t\delta\right) + \mathbb{P}(S_{\lfloor Ct \rfloor} \leq t).$$

Now, for $s \leq t$ and $k \in \mathbb{N}$, $\{M_t - M_s > k\} = \{S_{M_s+k} \leq t\}$. We define $(\hat{S}_k = S_{M_s+k} - S_{M_s})_{k\leq 1}$. It has the same law as $(S_k)_{k\leq 1}$. Then

$$\{S_{M_s+k} \leq t\} = \{S_{M_s} + \hat{S}_k \leq t\} \subset \{\hat{S}_k \leq t - s\}.$$

Then

$$\mathbb{P}\left(M_t - M_t^\varepsilon > t\delta\right) \leq \frac{(Ct^2)}{2}\mathbb{P}\left(S_{\lfloor t\delta \rfloor} \leq Ct\varepsilon\right) + \mathbb{P}(S_{\lfloor Ct \rfloor} \leq t).$$

Therefore

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\limsup_{t\to+\infty}\frac{1}{t}\ln\mathbb{P}\left(|M_t - M_t^\varepsilon| > t\delta\right) \\
&\quad\leq \limsup_{C\to\infty}\lim_{\varepsilon\to 0}\limsup_{t\to+\infty}\frac{1}{t}\ln\left[2\max\left\{\frac{(Ct^2)}{2}\mathbb{P}\left(S_{\lfloor t\delta \rfloor} \leq Ct\varepsilon\right), \mathbb{P}(S_{\lfloor Ct \rfloor} \leq t)\right\}\right] \\
&\quad\leq \max\left\{\lim_{\varepsilon\to 0}\limsup_{t\to+\infty}\frac{1}{t}\ln\mathbb{P}\left(S_{\lfloor t\delta \rfloor} \leq t\varepsilon\right), \limsup_{C\to\infty}\limsup_{t\to+\infty}\frac{1}{t}\ln\mathbb{P}(S_{\lfloor Ct \rfloor} \leq t)\right\}.
\end{aligned}$$

By the Markov inequality, we deduce

$$\mathbb{P}\left(S_{\lfloor t\delta \rfloor} \leq t\varepsilon\right) = \mathbb{P}\left(\mathrm{e}^{-S_{\lfloor t\delta \rfloor}/\varepsilon} \leq \mathrm{e}^{-t}\right) \leq \mathrm{e}^{t+\lfloor t\delta \rfloor \ln\mathbb{E}(\mathrm{e}^{-\tau_1/\varepsilon})}.$$

Finally

$$\lim_{\varepsilon\to 0}\limsup_{t\to+\infty}\frac{1}{t}\ln\mathbb{P}\left(S_{\lfloor t\delta \rfloor} \leq t\varepsilon\right) \leq \lim_{\varepsilon\to 0}\limsup_{t\to+\infty}\frac{1}{t}\left(t + \lfloor t\delta \rfloor \ln\mathbb{E}(\mathrm{e}^{-\tau_1/\varepsilon})\right) = -\infty$$

and

$$\limsup_{C\to\infty}\limsup_{t\to+\infty}\frac{1}{t}\ln\mathbb{P}(S_{\lfloor Ct \rfloor} \leq t) \leq \limsup_{C\to\infty}\limsup_{t\to+\infty}\frac{1}{t}\left(t + \lfloor tC \rfloor \ln\mathbb{E}(\mathrm{e}^{-\tau_1}\right) = -\infty.$$

□

Lemma 2.A.14. $\mu_t^{\varepsilon,n}(\varphi^\varepsilon)$ is an exponentially good approximation of Z_t^n/t, i.e. for all $\delta > 0$ it holds

$$\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\ln\mathbb{P}\left(\left|\mu_t^{\varepsilon,n}(\varphi^\varepsilon) - \frac{Z_t^n}{t}\right| > \delta\right) = -\infty.$$

Proof. In fact, we have

$$\left|\mu_t^{\varepsilon,n}(\varphi^\varepsilon) - \frac{Z_t^n}{t}\right| \leq \frac{n}{t}\left((M_t - M_t^\varepsilon) + 1\right),$$

thus for $\delta > 0$,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\ln\mathbb{P}\left(\left|\mu_t^{\varepsilon,n}(\varphi^\varepsilon) - \frac{Z_t^n}{t}\right| > \delta\right)\\ &\quad\leq \lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\ln\mathbb{P}\left(M_t - M_t^\varepsilon > \frac{\delta t}{n} - 1\right) = -\infty.\end{aligned}$$

□

2.B Complementary proofs

2.B.1 Relative entropy

Lemma 2.B.1. Let ν_1 and ν_2 two measures of the space $\mathcal{X}$, and ν_1 absolutely continuous with respect to ν_2. Let $(B_m)_{m\leq M}$ a partition of $\mathcal{X}$:

$$H(\nu_1|\nu_2) \geq \sum_{m=1}^{M}\nu_1(B_m)\ln\left(\frac{\nu_1(B_m)}{\nu_2(B_m)}\right).$$

Proof. By the properties of the relative entropy:

$$H(\nu_1|\nu_2) = \sup_f\left\{\int f d\nu_1 - \ln\int \mathrm{e}^f\, d\nu_2\right\},$$

where the supremum is taken over all f that are measurable and bounded.

For

$$f = \sum_{m=1}^{M}\ln\left(\frac{\nu_1(B_m)}{\nu_2(B_m)}\right)\mathbb{1}_{B_m},$$

we have:

$$\begin{aligned}\int \mathrm{e}^{\sum_{m=1}^M \ln\left(\frac{\nu_1(B_m)}{\nu_2(B_m)}\right)\mathbb{1}_{B_m}}\, d\nu_2 &= \sum_{n=1}^{M}\int_{B_n}\mathrm{e}^{\sum_{m=1}^M \ln\left(\frac{\nu_1(B_m)}{\nu_2(B_m)}\right)\mathbb{1}_{B_m}}\, d\nu_2\\ &= \sum_{n=1}^{M}\int_{B_n}\mathrm{e}^{\ln\left(\frac{\nu_1(B_n)}{\nu_2(B_n)}\right)}\, d\nu_2\\ &= \sum_{n=1}^{M}\left(\frac{\nu_1(B_n)}{\nu_2(B_n)}\right)\nu_2(B_n) = 1\end{aligned}$$

$$\begin{aligned}\int f d\nu_1 - \ln\int \mathrm{e}^f\, d\nu_2 &= \sum_{m=1}^{M}\ln\left(\frac{\nu_1(B_m)}{\nu_2(B_m)}\right)\int \mathbb{1}_{B_m} d\nu_1 - \ln\int \mathrm{e}^{\sum_{m=1}^M \ln\left(\frac{\nu_1(B_m)}{\nu_2(B_m)}\right)\mathbb{1}_{B_m}}\, d\nu_2\\ &= \sum_{m=1}^{M}\ln\left(\frac{\nu_1(B_m)}{\nu_2(B_m)}\right)\nu_1(B_m)\end{aligned}$$

Then:

$$H(\nu_1|\nu_2) \geq \sum_{m=1}^{M} \nu_1(B_m) \ln\left(\frac{\nu_1(B_m)}{\nu_2(B_m)}\right).$$

□

2.B.2 Lower semi-continuity

Lemma 2.B.2. Let $\mathcal{E}$ be a closed subset of $\mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$. Let f be a bounded lower semi-continuous function on $(0,+\infty]^2 \times \{1,...,n\}$. Let $R > 0$. Let Φ defined on $\mathcal{M}^1((0,+\infty]^2)$ by:

$$\Phi(\mu) = \begin{cases} \mu(f) \wedge R & \text{if } \mu \in \mathcal{E} \\ R & \text{else.} \end{cases}$$

Then Φ is lower semi-continuous.

Proof. We have to prove that for any $\mu_0 \in \mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$, for any sequence $(\mu_k)_k$ such that $\mu_k \rightharpoonup \mu_0$, we have: $\liminf \Phi(\mu_k) \geq \Phi(\mu_0)$.

Let show that $\mu \in \mathcal{E} \mapsto \mu(f)$ is lower semi-continuous.
We remind that for any positive function g on $(0,+\infty]^2 \times \{1,...,n\}$, we have :

$$\mu(g) = \int_0^{+\infty} \mu(\{g > t\})dt.$$

f is bounded: let A such that $\forall (x,y), f(x,y) \geq -A$. Let $g = f + A$, g is a positive lower semi-continuous function.
Let $\mu_0 \in \mathcal{E}$ and $(\mu_k)_k \in \mathcal{E}$ such that $\mu_k \rightharpoonup \mu_0$. Then, by Fatou's lemma:

$$\begin{aligned} \liminf \mu_k(g) &= \liminf \int_0^{+\infty} \mu_k(\{g > t\})dt \\ &\geq \int_0^{+\infty} \liminf \mu_k(\{g > t\})dt \end{aligned}$$

But $\{(x,y), g(x,y) > t\}$ is an open set because g is lower semi-continuous. By Portmanteau theorem, we have $\liminf \mu_k(\{g > t\}) \geq \mu_0(\{g > t\})$. Then:

$$\begin{aligned} \liminf \mu_k(g) &\geq \int_0^{\infty} \mu_0(\{g > t\})dt \\ &= \mu_0(g) \end{aligned}$$

And we have: $\liminf \mu_k(f) = \liminf \mu_k(g - A) = \liminf \mu_k(g) - A \geq \mu_0(g) - A = \mu_0(f)$. Then $\mu \mapsto \mu(f)$ is lower semi-continuous on $\mathcal{E}$.

Now, we show that Φ is lower semi-continuous. Let $\mu_0 \in \mathcal{M}^1((0,+\infty]^2 \times \{1,...,n\})$. Firstly, if $\mu_0 \notin \mathcal{E}$: $\mathcal{E}$ is a closed set, so $\mathcal{E}^c$ is an open set. We can find a neighborhood U of μ_0 included in $\mathcal{E}^c$. Then, for all $\mu \in U, \Phi(\mu) = R$. So, for every $t < R$, for all

$\mu \in U, \Phi(\mu) = R > t$.
Secondly, if $\mu_0 \in \mathring{\mathcal{E}}$, then $\Phi(\mu_0) = \mu_0(f) \wedge R$. We have already proved that $\mu \mapsto \mu(f)$ is lower semi-continuous. Then $\mu \mapsto \mu(f) \wedge R$ is lower semi-continuous. Indeed:

— if $\mu_0(f) \leq R$, then $\Phi(\mu_0) = \mu_0(f)$. Then for every $t < \mu_0(f)$, there exists a neighborhood U of μ_0 such that $\forall \nu \in U, \nu(f) > t$. As $t < \mu_0(f) \leq R$, $\Phi(\nu) = \nu(f) \wedge R > t$.

— if $\mu_0(f) > R$, then $\Phi(\mu_0) = R$. Then for every $t < \mu_0(f)$, there exists a neighborhood U of μ_0 such that $\forall \nu \in U, \nu(f) > t$. In particular, for $t = R$, $\forall \nu \in U, \nu(f) > R$ and $\Phi(\nu) = R$. Then for every $s < R = \Phi(\mu_0)$, $\forall \nu \in U, \Phi(\nu) = R > s$.

Eventually, if $\mu_0 \in \mathcal{E} \setminus \mathring{\mathcal{E}}$: let $t < \Phi(\mu_0)$. In particular, $R > t$. There exists a neighborhood U_1 of μ_0 in $\mathcal{E}$ such that: $\forall \nu \in U_1, \Phi(\nu) > t$. Let U be a neighborhood of μ_0 in $\mathcal{M}^1((0, +\infty]^2)$ such that $U \cap \mathcal{E} \subset U_1$. For all $\nu \in U \cap \mathcal{E}^c$, $\Phi(\nu) = R > t$, and for all $\nu \in U \cap \mathcal{E}, \Phi(\nu) > t$.
So Φ is a lower semi-continuous function. $\square$

Chapter 3

Limit theorems for Hawkes processes including inhibition.

This chapter consists in the article [20], accepted for publication in *Stochastic Processes and their Applications*. Notice that the companion paper [19] cited several times is the Chapter 2.

Abstract

In this paper we consider some non linear Hawkes processes with signed reproduction function (or memory kernel) thus exhibiting both self-excitation and inhibition. We provide a Law of Large Numbers, a Central Limit Theorem and large deviation results, as time growths to infinity. The proofs lie on a renewal structure for these processes introduced in [22] which leads to a comparison with cumulative processes. Explicit computations are made on some examples. Similar results have been obtained in the literature for self-exciting Hawkes processes only.

Contents

3.1 Introduction.

Hawkes processes have been introduced by Hawkes [43] and are widely used for modeling purposes: originally as models for the appearances of earthquakes [43, 44], but now in finance [42, 3] and econometrics or in neuroscience as models of spike trains of neurons [58, 41]. We refer to the bibliography of our references for more details.

A Hawkes process $t \mapsto N^h_t = N^h([0,t])$ is a point process on the real line $\mathbb{R}$ characterized by its initial condition on $]-\infty, 0]$ and its intensity process $t \mapsto \Lambda(t)$ through the infinitesimal relation

$$\mathbb{P}(N^h_{\cdot} \text{ has a jump in }]t, t+dt[|\mathcal{F}_t) \;=\; \Lambda(t)\,dt\,,$$

where $\mathcal{F}_t = \sigma(N^h(]-\infty, s[\,;\, s \leq t))$ is the natural filtration of the process and

$$\Lambda(t) \;=\; f\left(\lambda + \int_{]-\infty,t[} h(t-s)\,N^h(ds)\right)\,. \tag{3.1.1}$$

Here $\lambda \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}^+$ is the *jump rate function* and $h : \mathbb{R}^+ \to \mathbb{R}$ is the *reproduction function* (or *memory kernel*). We shall give a more precise definition in the next section (in particular on what happens before time 0) as well as results on existence and stability.

When f is linear or affine, the process is said to be linear. In this case one has to assume that $\lambda \geq 0$ and $h \geq 0$ too. Note that when h vanishes identically we recover a standard Poisson process. Otherwise the Hawkes process is called non linear. Actually, except for the behaviour of the shifted process ([62, 22]), very few papers are dealing with possibly negative or signed h. The negative part of h can be interpreted as self-inhibition.

It is very natural to look at the large time behaviour of $N^h_{\cdot}$, in particular the Law of Large Numbers (LLN) the Central Limit Theorem (CLT) and the deviations from the asymptotic mean or more generally the large deviations (LD).

In the linear case (recall that h is thus assumed to be non-negative) and assuming that $\parallel h \parallel_{L^1(du)} < 1$, both the LLN

$$\frac{N^h_t}{t} \to \frac{\lambda}{1- \parallel h \parallel_{L^1(du)}} := \mu \quad a.s. \text{ as } t \to +\infty\,, \tag{3.1.2}$$

and the CLT

$$\frac{N^h_t - \mu t}{\sqrt{t}} \Longrightarrow \mathcal{N}^h(0,\sigma^2) \quad \text{ with } \sigma^2 \;=\; \frac{\lambda}{(1- \parallel h \parallel_{L^1(du)})^3}\,,$$

where the convergence holds in distribution, have been shown (see e.g. [24]). Actually Bacry et al. [2] have obtained the functional version of the CLT (convergence to some Brownian motion) in the multivariate case. In a different direction, [36] have shown a CLT for fixed t as $\lambda \to +\infty$.

The easiest way to derive LLN and CLT in the linear case is presumably to use the immigration-birth representation also called the cluster process representation in [44], connecting N^h to subcritical Galton-Watson processes. This representation was used in [8] in order to get the Large Deviation (LD) principle for N_t^h/t with rate function

$$I(x) = x \ln \left(\frac{x}{\lambda + x \parallel h \parallel_{L^1(du)}} \right) - x(1 - \parallel h \parallel_{L^1(du)}) + \lambda \, .$$

For this explicit expression of the rate function see [85] p.761. The LD principle is obtained in [8] under the additional assumption $\int_0^{+\infty} t\, h(t)\, dt < +\infty$. It is claimed in the introduction of [35] that this assumption is not necessary. Under more restrictive assumptions, [35] contains precise deviations (see e.g. Theorem 2 therein).

The non linear case is of course more difficult. According to the general seminal paper by Brémaud and Massoulié [15], if f is L-Lipschitz and $L \parallel h \parallel_{L^1(du)} < 1$, there exists a unique stationary version of the Hawkes process. Rate of convergence to equilibrium is studied in [14] in two specific cases. As a consequence of Brémaud and Massoulié result, we get that

$$\frac{N_t^h}{t} \to \mu = \mathbb{E}_s[N^h([0,1])] \quad a.s. \text{ as } t \to +\infty \, , \tag{3.1.3}$$

where $\mathbb{E}_s$ denotes the expectation w.r.t. the stationary ergodic distribution.

In the particular situation where h is an exponential, the Hawkes process becomes Markovian and some results of large deviation have been obtained [86]. In [85], Zhu proved a functional CLT *at equilibrium* from which the following follows

Theorem 3.1.1. Assume that

(1) f is L-Lipschitz,

(2) h is non-negative, decreasing and such that $\int_0^{+\infty} t\, h(t)\, dt < +\infty$,

(3) $L \int_0^{+\infty} h(t)\, dt < 1$,

(4) $\lambda \geq 0$.

Then the stationary Hawkes process satisfies $\frac{N_t^h - \mu t}{\sqrt{t}} \Longrightarrow \mathcal{N}^h(0, \sigma^2)$ as $t \to +\infty$ in distribution, with

$$\sigma^2 := \mathrm{Var}_s(N^h([0,1])) + 2 \sum_{j \geq 1} \mathrm{Cov}_s(N^h([0,1]), N^h([j, j+1]))$$

where Var_s and Cov_s denote the variance and covariance w.r.t. the stationary distribution.

The proof is based on martingales techniques for the functional CLT. As the author himself is saying, to obtain an explicit expression for μ and σ^2 can rapidly become a difficult task. In the same work, Zhu also obtained a Strassen iterated logarithm law. One can also mention [87] where a large deviation result is obtained by contracting the level-3 LDP, i.e. by considering the shifted occupation measure. Theorem 2 in [87] then furnishes a LDP for N_t^h/t, provided h is non decreasing and non-negative and f is sub-linear at infinity. The expression of the rate function, as the infimum of the entropy on some set of measures satisfying a linear constraint is however not really tractable.

Since we are interested in neurosciences, our goal in this work is to understand the role of self-inhibition in the asymptotic behaviour of Hawkes processes. Since inhibition will slow down the neuronal activity, we thus have to consider signed functions h (the positive part modeling the self-excitation), but also jump rate functions f satisfying $f(u) = 0$ if $u \leq 0$. In the present paper, we will study the case of a general, signed, reproduction function with compact support and the specific jump rate function $f(u) = u^+ = \max(u, 0)$. This choice is of course the simplest one allowing us to introduce inhibition, and to compare this situation with linear models.

We will obtain a LLN, a CLT and deviation inequalities, where the parameters are characterized by the renewal structure of the process introduced in [22] replacing the classical cluster representation of the self-exciting case established in [44] which is no more valid. This renewal structure allows us to write the Hawkes process almost as a cumulative process.

The main tools are then limit theorems for cumulative processes and actually, the technical work consists in showing that one can apply these theorems in the present situation. An important tool is a comparison between the considered Hawkes process, the self excited process associated to the positive part of the reproduction function, furnishing an upper bound, and a purely inhibited process corresponding to the (negative) lower bound of the reproduction function (see Proposition 3.2.4), furnishing a lower bound.

For simplicity we restrict ourselves to an empty initial condition (see below). Some explicit computations are done in simple particular cases of pure inhibition (h non-positive). Precise statements will require some definitions, so that they are postponed to the next section. We emphasize, that the inhibition part introduces new intricacies.

As we said, very few papers are dealing with inhibition. In [27] some specific kernels are considered, but the addressed problem is not the one we are considering here. Looking at possibly negative reproduction functions is not only of mathematical interest. As shown in [41, 72, 21] a multivalued version of the model we are studying is particularly well suited for modeling spike train of neurons, at least in an almost stationary regime. To extend our results to the multivalued framework should thus be an interesting question.

3.2 Notation, definitions and results.

3.2.1 Hawkes processes

We consider an appropriate filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual assumptions.

Definition 3.2.1. Let $\lambda > 0$ and $h : (0, +\infty) \to \mathbb{R}$ a signed measurable function. Let N^0 a locally finite point process on $(-\infty, 0]$ with law $\mathbf{m}$.
The point process N^h on $\mathbb{R}$ is a Hawkes process on $(0, +\infty)$, with initial condition N^0 and reproduction measure $\mu(dt) = h(t)dt$ if:
- $N^h |_{(-\infty,0]} = N^0$,
- the conditional intensity measure of $N^h |_{(0,+\infty)}$ with respect to $(\mathcal{F}_t)_{t\geq 0}$ is absolutely continuous w.r.t the Lebesgue measure and has stochastic intensity:

$$\Lambda^h : t \in (0, +\infty) \mapsto \left(\lambda + \int_{(-\infty,t)} h(t-u)N^h(du)\right)^+. \tag{3.2.1}$$

where $x^+ = \max(x, 0)$.

The next proposition gives an explicit representation of the Hawkes process as solution of an SDE driven by a Poisson point process and states an important coupling property.

Proposition 3.2.2 (Proposition 2.1 in [22]). Let Q be a $(\mathcal{F}_t)_{t\geq 0}$ - two-dimensional Poisson point process on $(0, +\infty) \times (0, +\infty)$ with unit intensity. We consider the equation

$$\begin{cases} N^h = N^0 + \int_{(0,+\infty)\times(0,+\infty)} \delta_u \mathbb{1}_{\theta \leq \Lambda^h(u)} Q(du, d\theta) \\ \Lambda^h(u) = \left(\lambda + \int_{(-\infty,u)} h(u-s)N^h(ds)\right)^+, \ u > 0, \end{cases} \tag{3.2.2}$$

where $\lambda > 0$ is an immigration rate, $h : (0, +\infty) \to \mathbb{R}$ is a signed measurable function and N^0 is an initial condition of law $\mathbf{m}$ on $(-\infty, 0]$.
We consider the similar equation for N^{h^+} in which h is replaced by $h^+(.) = max(h(.), 0)$.
We assume that $\|h^+\|_1 :=\| h^+ \|_{L^1(du)} < 1$ and that the distribution $\mathbf{m}$ satisfies:

$$\forall t > 0, \int_0^t \mathbb{E}_{\mathbf{m}} \left(\int_{(-\infty,0]} h^+(u-s)N^0(ds)\right) du < +\infty. \tag{3.2.3}$$

Then:
- There exists a pathwise strong solution N^h of equation (3.2.2), and this solution is a Hawkes process.
- This property is true for N^{h^+}. Moreover, in the sense of measures, $N^h \leq N^{h^+}$, meaning that for all $0 \leq s \leq t < +\infty$, $N^h([s,t]) \leq N^{h^+}([s,t])$.

3.2.2 Definitions and assumptions

In this paper we consider a Hawkes process N^h according to Definition 3.2.1. We focus on the case of a signed reproduction function h which represents a possible inhibition on the appearance of future points.

Assumption 3.2.3. In all the paper, we will make the following assumptions :

i) $h : (0, +\infty) \to \mathbb{R}$ is a compactly supported signed measurable function. We define $L(h)$ as the supremum of the support of h: $L(h) := \sup\{t > 0, |h(t)| > 0\} < \infty$.

ii)

$$\|h^+\|_1 := \int_0^{+\infty} h^+(u)\, du < 1,$$

where $h^+(x) = max(h(x), 0)$.

iii) $\lambda > 0$,

iv) the initial condition on $]-\infty, 0[$ does not contain any point i.e. $\mathbf{m} = \delta_\emptyset$.

We are interested in the asymptotic behaviour of the number of jumps of the process N^h on the interval $[0, t]$, and we denote:

$$N_t^h = N^h([0, t]), \quad \forall t \geq 0$$

In particular we aim at quantifying precisely the loss of points due to inhibition. We will prove asymptotic results for $\frac{N_t^h}{t}$ and give exact computations on specific examples.

First we show another comparison result, this time furnishing a lower bound for N_t^h. This result motivates the detailed study of the canceling of intensity example.

Proposition 3.2.4 (Minoration of Hawkes process).
Let h be a function satisfying Assumptions 3.2.3. Let $\lambda > 0$ and define $g = -\lambda \mathbb{1}_{[0, L(h)]}$.
One can find a coupling of two Hawkes processes N^h and N^g, respectively associated with the reproduction functions h and g and with basal intensity λ, such that for any $t \geq 0$:

$$N_t^h \geq N_t^g \quad a.s.$$

Note that this comparison result is weaker than the majoration via h^+, since we do not have $N^h([s, t]) \geq N^g([s, t])$ for all s, but only for $s = 0$.

Proof. The main idea is to construct these two processes with the same Poisson point process Q on $(0, +\infty)^2$. We consider the successive jumps of N^h: $U_1^h, U_2^h, U_3^h, ...$; and the ones of N^g: $U_1^g, U_2^g, U_3^g, ...$.

We will prove by induction, that

$$\forall j \geq 1, N^h_{U_j^g} \geq N^g_{U_j^g} = j \quad \text{a.s.}$$

by studying the intervals associated with the $[U_j^g, U_{j+1}^g)$ for $j \in \mathbb{N}$. We stress out that considering the definition of the function g, the intensity Λ^g of the Hawkes process N^g can only take the two values 0 and λ.
<u>First interval:</u> First remark that $\forall t < \min(U_1^h, U_1^g)$,

$$\Lambda^h(t) = \lambda = \Lambda^g(t),$$

thus we have $U_1^h = U_1^g$ and consequently $N^h_{U_1^g} = N^g_{U_1^g}$.

Second interval: For $j = 2$: by definition, there is only one jump for N^g on $[U_1^g, U_2^g)$. There are two possibilities for N^h:

— Assume that there is no other jump that U_1^h in this interval. Since $U_2^g \geq U_1^g + L(h)$, we have $\Lambda^g(U_2^g-) = \lambda = \Lambda^h(U_2^g-)$. Accordingly, $U_2^h = U_2^g$ and in particular, $N^h_{U_2^g} = N^g_{U_2^g}$ a.s.

— Otherwise, there is at least one other jump of N^h in $(U_1^g, U_1^g + L(h))$. In this case, $N^h_{U_2^g} \geq 2 = N^g_{U_2^g}$ a.s.

Recursion step: We fix j and we suppose that the statement holds for $i \leq j$. Let $k = N^h_{U_j^g} \geq j$ by assumption. Then consider the two following cases:

— If U_j^g is a jump of N^h, there is either at least one other jump of N^h in (U_j^g, U_{j+1}^g) or no other jump. If there is no other jump, then $\Lambda^h(U_{j+1}^g-) = \Lambda^g(U_{j+1}^g-) = \lambda$, since $U_{j+1}^g > U_j^g + L(h)$. So, $U_{k+1}^h = U_{j+1}^g$. In both situations,

$$N^h_{U_{j+1}^g} \geq 1 + N^h_{U_j^g} \geq 1 + N^g_{U_j^g} = N^g_{U_{j+1}^g}.$$

— If U_j^g is not a jump of N^h, then

$$\Lambda^h(U_j^g-) < \lambda = \Lambda^g(U_j^g-).$$

Therefore since the support of h is of length $L(h)$ we deduce that

$$U_k^h < U_j^g < U_k^h + L(h).$$

By the induction hypothesis, we know that $k \geq j$. Then, there is either at least one jump of N^h in $(U_j^g, U_j^g + L(h))$, or the next jump is U_{j+1}^g, i.e. $U_{k+1}^h = U_{j+1}^g$. In both cases, we have $N^h_{U_{j+1}^g} \geq 1 + k \geq 1 + j = N^g_{U_{j+1}^g}$.

This concludes the induction. Let us come back to a general $t \in \mathbb{R}^+$. For any fixed ω, there exists $j = j(\omega) \in \mathbb{N}$, such that: $U_j^g(\omega) \leq t < U_{j+1}^g(\omega)$. Then using the monotonicity of N^h we have

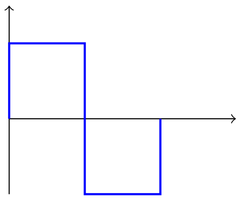
$$N_t^h \geq N^h_{U_j^g} \geq N^g_{U_j^g} = N_t^g.$$

□

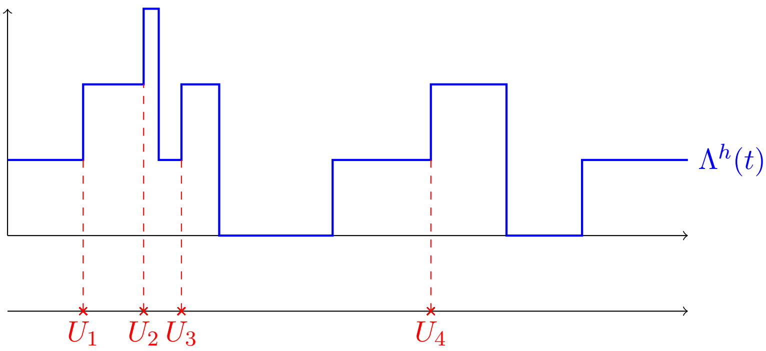
Both comparison results may be used in the sequel.

3.2.3 Hawkes processes as cumulative processes.

Our study of the large time behaviour of Hawkes processes lies on a renewal structure for Hawkes processes first introduced in [22] we shall partly recall below. Notice that this structure is used in [22] for a completely different purpose.



(a) Example of function h: a square wave



(b) Example of simulation of a Hawkes process with parameter h: in blue, the intensity function $t \mapsto \Lambda^h(t)$; in red, the jumps times. The axis below indicates the Dirac measures of the process.

Figure 3.1 – Example of Hawkes process

Let N^h be a Hawkes process according to Definition 3.2.1, with initial condition $N^0 = \emptyset$. We denote by $U_1, U_2, U_3, ...$ its successive jumps.

Let us introduce the renewal times of the process which splits the time line into independent and identically distributed time windows of length $\tau_1, \tau_2, \cdots$.

Define the stopping time

$$\tau_1 = \inf\{t > U_1, N^h((t - L(h), t]) = 0\},$$

that is the first time after U_1 such that there has been no jump during a time $L(h)$. We also set

$$S_0 = 0 \quad \text{and} \quad S_1 = \tau_1.$$

Let us now define

$$W_1 = N^h([U^1, S_1]) = N^h([0, S_1]),$$

the number of jumps of the process in this first time window and rename the jump times in the first time window as:

$$U_j^1 = U_j, \quad \forall j \in \{1, \cdots, W_1\}.$$

We shall see below that τ_1 and W_1 are almost surely finite. Recursively let $i \in \mathbb{N}^*$ such that $(\tau_1, W_1), ...(\tau_i, W_i)$ are well defined (and a.s. finite). Let $S_i = \sum_{k=1}^{i} \tau_k$ and define

$$U_1^{i+1} = U_{W_1 + ... + W_i + 1},$$

and

$$\tau_{i+1} = \inf\{t > U_1^{i+1}, N^h((t - L(h), t]) = 0\} - S_i, \tag{3.2.4}$$

Notice that there is at least one jump in $[S_i, S_i + \tau_{i+1}]$. We finally introduce the number of jumps in the $(i+1)$'th window as

$$W_{i+1} = N^h([U_1^{i+1}, S_i + \tau_{i+1}) = N^h([S_i, S_i + \tau_{i+1}]), \tag{3.2.5}$$

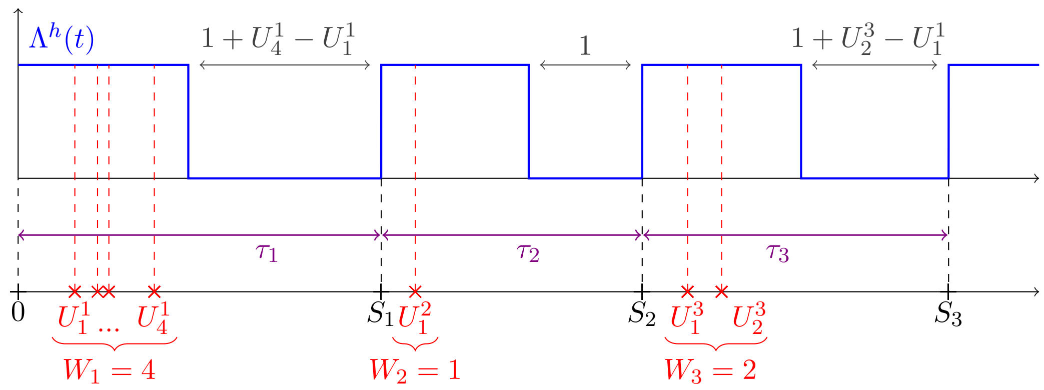


Figure 3.2 – Example of the evolution of intensity in function of time and renumbering of jumps in the case where $h = -\lambda \mathbb{1}_{[1,2]}$.

and rename the associated jump times as:

$$U_j^{i+1} = U_{W_1+\ldots+W_i+j}, \quad \forall j \in \{1, \ldots, W_{i+1}\}.$$

Figure 3.2 is an example of this splitting of the time and the renumbering of the jumps, in the case where $h(t) = -\lambda \mathbb{1}_{(1,2)}(t)$, so that $L(h) = 2$.

The next Proposition gathers important properties on the law of (τ_i, W_i) defined above. However more explicit information are difficult to obtain except in specific cases (see Section 3.3).

Proposition 3.2.5. Under Assumptions 3.2.3, and using the above definitions:

i) the $(\tau_i, W_i)_i$ are i.i.d. random variables,

ii) for $i \in \mathbb{N}^*$, the $(U_1^i - S_{i-1})$ are i.i.d. random variables with exponential distribution $\mathcal{E}(\lambda)$, that is, the time between the beginning of a window and the first point of this window follows an exponential law.

Proof. Let Q be a two-dimensional Poisson point process, and let N^h generated by Q as in proposition 3.2.2, the $(\tau_i, W_i)_i$ being defined as before.

Given τ_1, remark that U_1^2 is the first jump of Q on $(\tau_1, +\infty) \times [0, \infty]$. Indeed, using successively the definition of $L(h)$ and τ_1 we deduce that:

$$\begin{aligned}\Lambda(\tau_1) &= \left(\lambda + \int_{(-\infty,\tau_1)} h(t-u) N^h(du)\right)^+ \\ &= \left(\lambda + \int_{(\tau_1 - L(h),\tau_1)} h(t-u) N^h(du)\right)^+ \\ &= \lambda.\end{aligned}$$

By translation, $U_1^2 - \tau_1$ is the first jump of a Poisson point process Q' on $(0,\infty) \times [0,\infty]$, independent of Q on $(0,\tau_1) \times [0,\infty]$, and $U_1^2 - \tau_1$ is independent of $\tau_1 = S_1$.

Since the jumps of N^h before τ_1 do not influence $\Lambda^h(t)$ for $t > \tau_1$ (by definition of $L(h)$ and τ_1),

$$\tau_2 = \inf\{t > U_1^2 - S_1, N^h((t + S_1 - L(h), t + S_1]) = 0\},$$

only depends on Q on $(U_1^2, +\infty) \times [0, \infty]$. Moreover, $(0, S_1) \times [0, \infty]$ and $(U_1^2, +\infty) \times [0, \infty]$ are almost surely disjoints. Hence Q on $(U_1^2, +\infty) \times [0, \infty]$ is independent of Q on $(0, S_1) \times [0, \infty]$ so that τ_2 is independent of (τ_1, W_1).
The number of points in the second time window $W_2 = N^h([U_1^2, S_2]) = N^h([S_1, S_2])$ only depends on Q on $(U_1^2, +\infty) \times [0, \infty]$. W_1 depends on Q on $(0, S_1) \times [0, \infty]$. For the same reason as before, W_2 is independent of (τ_1, W_1). The same argument can be used for each k: as the $(S_k)_k$ split $\mathbb{R}^+$ in disjoints intervals, then Q on each of these intervals is independent of Q of another interval.

In particular, $U_1^1 = U_1^1 - S_0$, $U_1^2 - S_1$ (and all the following) are independent and can be defined as the first jump of a Poisson point process on $(0, +\infty) \times [0, \infty]$. Then they follows an exponential law of parameter λ.
Using time translation, we see that τ_1, τ_2 (and so on) are defined the same way and follow the same law. Then W_1, W_2 (and so on) are defined the same way and follow the same law. □

This construction indicates the renewal structure generated by the Hawkes process. We shall use this structure to prove limit theorems.

To this end remark that

$$N_t^h = \sum_{i=1}^{\infty} \mathbb{1}_{U_i \le t} = \sum_{i=1}^{\infty} \sum_{j=1}^{W_i} \mathbb{1}_{U_i^j \le t}. \tag{3.2.6}$$

Introduce the renewal process associated to the S_i's

$$M_t^h := \sum_{i=1}^{\infty} \mathbb{1}_{S_i \le t}. \tag{3.2.7}$$

Since $S_i = \sum_{k=1}^{i} \tau_k$ we may introduce

$$\hat{N}_t^h := \sum_{i=1}^{\infty} W_i \mathbb{1}_{S_i \le t} = \sum_{i=1}^{M_t^h} W_i. \tag{3.2.8}$$

For any $t \in \mathbb{R}^+$, the current window is the $M_t^h + 1$-th. $\hat{N}_t^h$ includes only the jumps up to the M_t^h-th window, while N_t^h can have more jumps. In particular,

$$\hat{N}_t^h \le N_t^h \le \hat{N}_t^h + W_{M_t^h+1} \quad \text{a.s.} \tag{3.2.9}$$

We thus have

$$N_t^h = \sum_{i=1}^{M_t^h} W_i + R_t^h$$

for some renewal process M_t^h and a remaining term $R_t^h \leq W_{M_t^h+1}$, the W_i's being i.i.d.. Such processes are known as *cumulative processes* in the literature. A LLN and a CLT for $\hat{N}_t^h$ can be found in [1] theorems 3.1 and 3.2. The LD principle for cumulative processes is studied in [52] in the special case

$$W_i = F(\tau_i)$$

for some non-negative, bounded and continuous function F (see the references in [52] for some previous results in still more specific cases). These results do not apply for Hawkes processes, and we had to establish a more general LD principle in the companion paper [20]. In order to get similar results for N_t^h/t it will remain to study the remaining R_t^h.

3.2.4 LLN, CLT and asymptotic deviations for signed reproduction function h.

We now state the main results of the paper. The key is to get enough moments for the (τ_i, W_i). The first result deals with this problem.

Proposition 3.2.6. Let h be a signed function satisfying Assumptions 3.2.3. Let us consider the Hawkes process N^h and the i.i.d. couples of random variables (τ_i, W_i) defined in (3.2.4)-(3.2.5).

i) For $\alpha < \alpha_0 := \min\left(\lambda\,,\ \frac{||h^+||_1 - \ln(||h^+||_1)-1}{L(h)}\right)$ we have $\mathbb{E}(e^{\alpha\tau_1}) < +\infty$.

ii) There exists $\theta_0 > 0$ such that for $\theta < \theta_0$, $\mathbb{E}(e^{\theta W_1}) < +\infty$.

In particular τ_1 and W_1 have polynomial moments of any order.

The proof of this proposition is given in Section 3.4.1. Actually, one can give a lower bound on θ_0. This lower bound differs whether $h \leq 0$ or not.
In the general case, the upper bound for θ_0 depends on a random variable $\mathcal{S}$ with distribution

$$\mathbb{P}(\mathcal{S} = k) = \frac{e^{-k\|h^+\|_1}(k\|h^+\|_1)^{k-1}}{k!}.$$

Using a comparison with a queue process that will be detailed in the proof of Proposition 3.2.6 below one prove that θ_0 can be chosen as

$$\theta_0 < ||h^+||_1 - \ln(||h^+||_1) - 1 \quad \text{and} \quad \lambda(\mathbb{E}(e^{2\theta_0\mathcal{S}}) - 1) < \alpha_0.$$

In the case of pure inhibition, i.e. $h \leq 0$, the quantity $||h^+||_1 - \ln(||h^+||_1)$ becomes infinite. However using a comparison with a Poisson Process one can get another explicit bound for θ_0, whose proof will also be given in Section 3.4.1.

Proposition 3.2.7. If $h \leq 0$, one can choose $\theta_0 < -\ln\left(1 - e^{-\lambda L(h)}\right)$ in Proposition 3.2.6.

Remark 3.2.8. Exacts computations for moments of τ and W are difficult. Let us consider here and in section 3.3 some specific cases.

Notice that for $h = 0$ (i.e. in the case of a Poisson process), $W_1 = 1$ has exponential moments of any order and τ_1 whose distribution is exponential with parameter λ, has exponential moments up to order λ.

Another basic case is the *canceling of intensity* case, i.e. choosing the reproduction function as $g = -\lambda \mathbb{1}_{[0,A]}$ for some positive λ and A. We have seen in Proposition 3.2.4 that the corresponding N_t^g is smaller than any N_t^h with $L(h) = A$. Since for $t \in (U_1^1, U_1^1 + A)$ it holds $\Lambda^h(t) = 0$, it immediately follows that $\tau_1 = U_1^1 + A$ and $W_1 = 1$, so that

$$(W_1, \tau_1) \sim (1, A + \mathcal{E}(\lambda)),$$

so that $\mathbb{E}(\tau_1) = A + \lambda^{-1}$, $\mathrm{Var}(\tau_1) = \lambda^{-2}$, $\alpha_0 = \lambda$ and $\theta_0 = +\infty$. ♢

From these moments properties and the renewal structure of the Hawkes process, we will derive the following asymptotic results:

Theorem 3.2.9 (**Law of Large Numbers**). Let h be a signed function satisfying Assumptions 3.2.3 and consider the Hawkes process N^h given by (3.2.2). Then we have the following:

$$\frac{N_t^h}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\mathbb{E}[W_1]}{\mathbb{E}(\tau_1)}.$$

Thanks to our comparison results and to (3.1.2) we have

$$\frac{\lambda}{1 + \lambda L(h)} \leq \frac{\mathbb{E}[W_1]}{\mathbb{E}(\tau_1)} \leq \frac{\lambda}{1 - ||h^+||_1}.$$

Our method will also provide us with a CLT.

Theorem 3.2.10 (**Central Limit Theorem**). Let h be a signed function satisfying Assumptions 3.2.3 and consider the Hawkes process N^h given by (3.2.2). Then

$$\sqrt{t}\left(\frac{N_t^h}{t} - \frac{\mathbb{E}[W_1]}{\mathbb{E}(\tau_1)}\right) \underset{t\to\infty}{\Longrightarrow} \mathcal{N}(0, \sigma^2)$$

with

$$\sigma^2 = \frac{\mathrm{Var}\left(W_1 - \tau_1 \frac{\mathbb{E}[W_1]}{\mathbb{E}(\tau_1)}\right)}{\mathbb{E}(\tau_1)}.$$

We finally state deviation results based on the results in the companion paper [20], in which we obtain large deviation for general cumulative processes. To this end we need to introduce some notations. We also refer to [25] for a more general introduction on the topic.

Definition 3.2.11. We introduce the Cramer transform for $(a, b) \in \mathbb{R}^2$,

$$\Lambda^*(a, b) = \sup_{x,y} \left\{ax + by - \ln\left(\mathbb{E}\left[e^{x\tau_1 + yW_1}\right]\right)\right\}.$$

We also define for $z \in \mathbb{R}^+$,

$$J(z) = \inf_{\beta>0} \left(\beta\, \Lambda^* \left(\frac{1}{\beta} \, , \, \frac{z}{\beta} \right) \right) .$$

Similarly we define Λ_n^* and J_n replacing W_1 by $\min(W_1, n)$. Finally define

$$\tilde{J}(z) = \sup_{\delta>0} \liminf_{n\to\infty} \inf_{|y-z|<\delta} J_n(y) \, .$$

Thanks to Proposition 3.2.6 we may apply Theorem 2.4 in [20], telling us that the distributions of $\hat{N}_t^h/t$ satisfy asymptotic deviation inequalities .

Theorem 3.2.12. Recall that θ_0 is defined in Proposition 3.2.6 (ii).

— If $\theta_0 = +\infty$, the laws of the family $\hat{N}_t^h/t$ satisfy a large deviation principle with rate function $\tilde{J}$, i.e

1. for any closed set $\mathcal{C} \in \mathbb{R}$,

$$\limsup_{t\to\infty} \frac{1}{t} \, \ln \mathbb{P} \left(\hat{N}_t^h/t \, \in \, \mathcal{C} \right) \, \leq \, - \, \inf_{m\in\mathcal{C}} \, \tilde{J}(m)$$

2. for any open set $\mathcal{O} \in \mathbb{R}$,

$$\liminf_{t\to\infty} \frac{1}{t} \, \ln \mathbb{P} \left(\hat{N}_t^h/t \, \in \, \mathcal{O} \right) \, \geq \, - \, \inf_{m\in\mathcal{O}} \, \tilde{J}(m) \, ,$$

— If $\theta_0 < +\infty$, denoting $m = \mathbb{E}(W_1)/\mathbb{E}(\tau_1)$ we have for all $a > 0$ and all $\kappa \in (0, 1)$

$$\limsup_{t\to+\infty} \frac{1}{t} \, \ln \mathbb{P} \left(\frac{\hat{N}_t^h}{t} \geq m + a \right) \leq - \, \min \left[\inf_{z\geq m+\kappa a} J(z) \, , \, (1-\kappa)\theta_0 a \right] \, ,$$

and

$$\limsup_{t\to+\infty} \frac{1}{t} \, \ln \mathbb{P} \left(\frac{\hat{N}_t^h}{t} \leq m - a \right) \leq - \, \min \left[\inf_{z\geq m-\kappa a} J(z) \, , \, (1-\kappa)\theta_0 a \right] \, .$$

The latter deviation inequalities are obtained using that $J \leq \tilde{J}$ (see [20]). The proof of Theorem 2.4 in [20] is inspired by the one in [52] and uses a contraction principle applied to the Large Deviation Principle (LDP for short) at the level of the empirical process. The full LDP is obtained when all exponential moments exist. Otherwise we mainly obtain the upper bound, which is enough to get asymptotic deviations results, as explained in the Theorem below.

Corollary 3.2.13. Recall that θ_0 is defined in Proposition 3.2.6 (ii).

(1) If $\theta_0 = +\infty$, N_t^h/t satisfies the same LDP as $\hat{N}_t^h/t$.

(2) If $\theta_0 < +\infty$, we have for all $a > 0$ and $\kappa \in (0,1)$

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{N_t^h}{t} \geq m+a\right) \leq -\min\left[\inf_{z-m\geq\kappa a} J(z)\,,\, \frac{1-\kappa}{2}\theta_0 a\right], \tag{3.2.10}$$

Similarly

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{N_t^h}{t} \leq m-a\right) \leq -\min\left[\inf_{m-z\leq\kappa a} J(z)\,,\, \frac{(1-\kappa)}{2}\theta_0 a\right]. \tag{3.2.11}$$

for $\kappa \in (0,1)$.

Remark 3.2.14. Once again we may get an explicit expression for the rate function in the canceling intensity case $h = -\lambda\,\mathbb{1}_{[0,A]}$. Since $W_1 = 1$ and $\tau_1 - A$ is an exponential variable with parameter λ, we have

$$\beta\Lambda^*\left(\frac{1}{\beta}, \frac{m}{\beta}\right) = \sup_{x,y}\left(x + (m-\beta)y + \beta\ln\left(1 - \frac{x}{\lambda}\right) - \beta A x\right).$$

Notice that for a given x, $\sup_y \left(x + (m-\beta)y + \beta\ln\left(1-\frac{x}{\lambda}\right) - \beta A x\right) < +\infty$ if and only if $\beta = m$ due to the linear term in y. We deduce

$$J(m) = m\,\Lambda^*\left(\frac{1}{m}, 1\right).$$

It easily follows

$$J(m) = \lambda(1-mA) - m + m\ln\left(\frac{m}{\lambda(1-mA)}\right).$$

♢

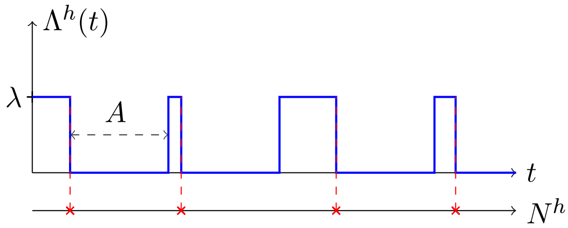
Remark 3.2.15. *Added to the proof.*

After completing the present paper and [20], another proof of a LDP for cumulative processes was proposed in the preprint [84]. Some of the results in [84] should possibly complete the picture when $\theta_0 < +\infty$. ♢

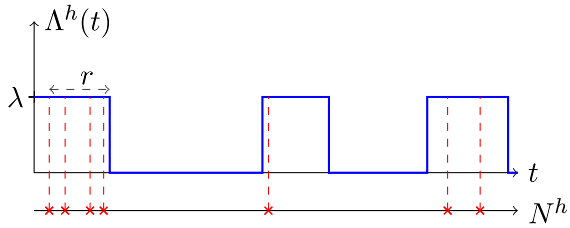
3.3 One more example with explicit calculations: canceling intensity with delay.

We already discussed in Remark 3.2.8 the *canceling of intensity* case $h = -\lambda\,\mathbb{1}_{[0,A]}$.

In our second example we add a delay to the previous case: the inhibition only occurs after a lag period of length $r > 0$. Let $\lambda > 0$, $r > 0$ and $A > r$ we consider $h = -\lambda\mathbb{1}_{[r,r+A]}$. Then $L(h) = r + A$. We can again explicitly compute the law of W_i and τ_i.



(a) Example of Hawkes process : canceling intensity without delay, $h = -\lambda \mathbb{1}_{[0,A]}$.



(b) Example of Hawkes process : canceling intensity with a delay, $h = -\lambda \mathbb{1}_{[r,r+A]}$.

Figure 3.3 – Comparison of Hawkes processes with or without a delay in the canceling of the intensity:
In blue, the intensity function $t \mapsto \Lambda^h(t)$; in red, the jumps times. The axis below indicates the Dirac measures of the process.

We can summarize the results of this two cases and apply Theorem 3.2.9 to obtain

Proposition 3.3.1. Let us consider $A > 0$ and $r \geq 0$. The Hawkes process associated with $h = -\lambda \mathbb{1}_{[r,r+A]}$ satisfies

$$\lim_{t\to\infty} \frac{N_t^h}{t} = \frac{\lambda(1+\lambda r)}{\lambda A + 2\lambda r + \mathrm{e}^{-r}} \quad a.s.$$

Remark 3.3.2. This result naturally leads to some comments on the issues brought by inhibition.

- Let us first remark that as $r \to 0$ we recover the result of the canceling intensity case given in Remark 3.2.8.
- Secondly we wonder whether one of both examples admits more points asymptotically. Therefore we are lead to study the ratio

$$\frac{\frac{\lambda}{\lambda A+1}}{\frac{\lambda(1+\lambda r)}{\lambda A+2\lambda r+e^{-\lambda r}}} = \frac{\lambda A + 2\lambda r + e^{-\lambda r}}{(\lambda A+1)(1+\lambda r)},$$

or equivalently the sign of

$$e^{-\lambda r} - 1 + \lambda r - \lambda^2 A r = \lambda^2 r\left(\frac{r}{2} - A\right) + \sum_{k=3}^{\infty} \frac{(-\lambda r)^k}{k!}.$$

using the series expansion of the exponential. We therefore deduce that since $A > r$, the right hand side is negative, and thus the ratio is less that 1. Consequently, this proves that the lag induces asymptotically more points in the inhibited Hawkes process. Note that even if our proof only holds in the case of $A > r$ since in this case there is a single time interval with canceled intensity between successive windows (see the proof), we expect that delay in inhibition will also induce more points in more general framework.

Notice finally that $||h||_1$ is the same in the delayed and the non delayed case, therefore we deduce that the asymptotic proportion of points is not fully characterized by $||h||_1$ and that formula (3.1.2) cannot be extended to the inhibited setting.

$\diamondsuit$

Proof of Proposition 3.3.1. Let us study Λ^h on the time interval $[U_1^1, U_1^1 + r + A]$:

— for $t \in [U_1^1, U_1^1 + r)$, then for any $u \in (0,t)$, $t-u$ belongs to $(0,t)$ and thus

$$\Lambda^h(t) = \left(\lambda + \int_0^t -\lambda \mathbb{1}_{(t-u)\in[r,r+A]} N^h(du)\right)^+ = \lambda,$$

— for $t \in [U_1^1 + r, U_1^1 + r + A]$, then

$$\Lambda^h(t) = \left(\lambda + \int_{(0,t)} -\lambda \mathbb{1}_{(t-u)\in[r,r+A]} N^h(du)\right)^+ \leq \left(\lambda - \lambda \mathbb{1}_{(t-U_1^1)\in[r,r+A]}\right)^+ = 0\,.$$

From this, we deduce that all the points of N^h in $]U_1^1, U_1^1 + r + A]$ actually belong to the interval $]U_1^1, U_1^1 + r[$. In particular, if N^h has no points in $]U_1^1, U_1^1 + r + A]$, then $W_1 = 1$ and $\tau_1 = U_1^1 + r + A$.
Let us now remark that $N^h(]U_1^1, U_1^1 + r])$ follows a Poisson law of parameter λr since the intensity of Hawkes process is constant on this interval. In particular $N^h([U_1^1, U_1^1 + r])$ is finite almost surely. More generally for any $1 < k \leq N^h([U_1^1, U_1^1 + r])$, then $U_k^1 \leq U_1^1 + r$ and

$$\forall t \in [U_k^1 + r, U_k^1 + r + A], \quad \Lambda^h(t) = 0.$$

Finally, since $A > r$ we have that $U_k^1 + r \leq U_1^1 + r + r \leq U_1^1 + r + A$, and thus the intensity remains null on the interval $[U_1^1 + r, U_k^1 + r + A]$.
We can conclude that

$$W_1 = N^h([U_1^1, U_1^1 + r]), \tag{3.3.1}$$
$$\tau_1 = U_{W_1}^1 + r + A. \tag{3.3.2}$$

Since the first point in N^h in the interval $[U_1^1, U_1^1 + r]$ is in U_1^1 we actually have

$$W_1 = 1 + N^h((U_1^1, U_1^1 + r)).$$

It follows that

$$W_1 - 1 \sim \mathcal{P}(\lambda r)$$

and

$$\mathbb{E}(W_1) = 1 + \lambda r\,, \quad \mathrm{Var}(W_1) = \lambda r \quad \text{and} \quad \theta_0 = +\infty\,.$$

We finally study the law of τ_1. From Equation (3.3.2), we can write

$$\begin{aligned}\tau_1 &= U_{W_1}^1 + r + A \\ &= r + A + U_1^1 + (U_{W_1}^1 - U_1^1),\end{aligned}$$

where $U_1^1 \sim \mathcal{E}(\lambda)$ by lemma 3.2.5 and U_1^1 and $(U_{W_1}^1 - U_1^1)$ are independent.
It remains to study the law of $(U_{W_1}^1 - U_1^1)$.

Thanks to (3.3.1), $0 \leq U_{W_1}^1 - U_1^1 \leq r$.

Let $t \in [0, r]$, we have:

$$\mathbb{P}\left(0 \leq U_{W_1}^1 - U_1^1 \leq t\right) = \sum_{k=1}^{\infty} \mathbb{P}\left(\{W_1 = k\} \cap \{U_{W_1}^1 - U_1^1 \leq t\}\right).$$

For $k = 1$: $\mathbb{P}\left(\{W_1 = 1\} \cap \{U_{W_1}^1 - U_1^1 \leq t\}\right) = \mathbb{P}(W_1 = 1) = \mathrm{e}^{-\lambda r}$.

For $k > 1$ since the intensity of the Hawkes process remains constant equal to λ on $[U_1^1, U_{W_1}^1]$ we can write

$$U_{W_1}^1 - U_1^1 \overset{(law)}{=} \sum_{k=1}^{W_1 - 1} T_k$$

where $(T_k)_{k\in\mathbb{N}}$ is a sequence of i.i.d $\mathcal{E}(\lambda)$. We can consider $(T_k)_{k\in\mathbb{N}}$ as the interarrival times of a Poisson process of parameter λ coupled with our Hawkes process, as in Proposition 3.2.2. Then, $T_0 = U_1^1$, and $(T_k)_{k\geq W_1}$ are defined. Then:

$$\begin{aligned}
\mathbb{P}\left(\{W_1 = k\} \cap\{U_{W_1}^1 - U_1^1 \leq t\}\right) &= \mathbb{P}\left(\left\{0 \leq \sum_{i=1}^{k-1} T_i \leq t\right\} \cap \left\{T_k + \sum_{i=1}^{k-1} T_i > r\right\}\right) \\
&= \mathbb{E}\left[\mathbb{1}_{0\leq\sum_{i=1}^{k-1} T_i \leq t} \mathbb{P}\left(T_k + \sum_{i=1}^{k-1} T_i > r \mid (T_1, ..., T_{k-1})\right)\right] \\
&= \mathbb{E}\left[\mathbb{1}_{0\leq\sum_{i=1}^{k-1} T_i \leq t}\, \mathrm{e}^{-\lambda\left(r - \sum_{i=1}^{k-1} T_i\right)}\right] \\
&= \int_{(\mathbb{R}^+)^{k-1}} \mathbb{1}_{0\leq\sum_{i=1}^{k-1} s_i \leq t}\ \lambda^{k-1}\, \mathrm{e}^{-\lambda \sum_{i=1}^{k-1} s_i} \times \mathrm{e}^{-\lambda(r - \sum_{i=1}^{k-1} s_i)}\, ds_2...ds_k \\
&= e^{-\lambda r} \lambda^{k-1} I_{k-1}(t) \\
&= e^{-\lambda r} \frac{(\lambda t)^{k-1}}{(k-1)!}.
\end{aligned}$$

with

$$I_k(t) := \int_{(\mathbb{R}^+)^k} \mathbb{1}_{0\leq\sum_{i=1}^{k} s_i \leq t} ds_1...ds_k = \frac{t^k}{k!}.$$

Thus

$$\begin{aligned}
\mathbb{P}\left(0 \leq U_{W_1}^1 - U_1^1 \leq t\right) &= \mathrm{e}^{-\lambda r} + \sum_{k\geq 2} e^{-\lambda r} \frac{(\lambda t)^{k-1}}{(k-1)!} \\
&= \mathrm{e}^{-\lambda(r-t)}.
\end{aligned}$$

Hence the distribution of $U_{W_1}^1 - U_1^1$ is given by $\mathrm{e}^{-\lambda r}\, \delta_0 + \lambda\, \mathrm{e}^{-\lambda(r-t)}\, \mathbb{1}_{(0,r]}(t) dt$.

An easy computation gives $\mathbb{E}(U^1_{W_1} - U^1_1) = r - \frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda r})$. Finally we obtain that

$$\begin{aligned}\mathbb{E}(\tau_1) &= r + A + \mathbb{E}(U^1_1) + \mathbb{E}(U^1_{W_1} - U^1_1)\\ &= r + A + \lambda^{-1} + r - \lambda^{-1}(1 - \mathrm{e}^{-\lambda r})\\ &= 2r + A + \lambda^{-1}\,\mathrm{e}^{-\lambda r}\end{aligned}$$

From Theorem 3.2.9 we obtain the following LLN

$$\frac{N^h_t}{t} \xrightarrow[t\to\infty]{a.s.} \frac{1+\lambda r}{2r + A + \lambda^{-1}\,\mathrm{e}^{-\lambda r}} = \frac{\lambda(1+\lambda r)}{\lambda A + 1 + (\mathrm{e}^{-\lambda r} - 1) + 2\lambda r}.$$

□

3.4 Proofs.

3.4.1 Proofs of Proposition 3.2.6 and 3.2.7

We start by proving that the random variables τ and W admit exponential moments.

Proof of Proposition 3.2.6.

Let h be a signed measurable function and h^+ its positive part. We generate N^h and N^{h^+} by coupling as in Proposition 3.2.2. Recall that $\|h^+\|_1 < 1$.

We denote by W_i, τ_i, S_i, ... (respectively W_i^+, τ_i^+, S_i^+, ...) the renewal quantities associated to N^h (resp. N^{h^+}). *Be careful that the previous construction of W_i^+, τ_i^+, S_i^+ is done by using intervals of length $L(h)$ not $L(h^+)$. Notice that since $L(h) \geq L(h^+)$, then the renewal structure is well defined for N^{h^+}. Moreover, if $h \leq 0$, $L(h^+) = 0$, one can replace h^+ by $h^+_\varepsilon = h^+ + \varepsilon\,\mathbb{1}_{[0,L(h)]}$ and then let ε go to 0 in order to compare with [22].*

Thanks to Proposition 3.2.2, we have $N^h \leq N^{h^+}$ a.s. We also know that $U^1_1 = U^{+,1}_1$.

Moreover, $\tau_1 \leq \tau_1^+$ a.s. because the jumps of N^h are included in those of N^{h^+}. We also have $W_1 = N^h([0,\tau_1]) \leq N^h([0,\tau_1^+]) \leq N^{h^+}([0,\tau_1^+]) = W_1^+$ a.s. So $W_1 \leq W_1^+ a.s.$

Study of N^{h^+}:
First, we focus on N^{h^+}. According to [22], we can associate a $M/G/\infty$ queue to N^{h^+}. To do this, we consider:

$$\Lambda^{h^+}(t) = \lambda + \int_{(-L(h),t]} h^+(t-u) N^{h^+}(du).$$

We can consider the Hawkes process as the sum of:

— the arrivals of ancestors V_k at rate λ and

— a continuous time Galton-Watson process beginning at each V_k with the following characteristics: the number of descendants follows a Poisson distribution with mean $\|h^+\|_1$ and the times of births have the density $h^+/\|h^+\|_1$.

In fact, to each arrival of an ancestor V_k, we can associate a time H_k corresponding to the life time of the cluster of V_k. V_k is independent of H_k and the $(H_k)_k$'s are independent.

We can associate to this process a queue in the following way:

— the customers are the ancestors and arrive at rate λ,

— the service time for each customer is $H_k + L(h)$.

We denote by Y_t the number of customers in the queue at time t:

$$Y_t = \sum_k \mathbb{1}_{V_k \le t < V_k + H_k + L(h)}.$$

Let $\mathcal{T}_1^+ = \inf\{t \ge 0, Y_{t-} \neq 0, Y_t = 0\}$, be the first time the queue is empty. By proposition 2.6 of [22], we have:

$$\forall \alpha < \alpha_0 := \min\left(\lambda, \frac{\|h^+\|_1 - \log(\|h^+\|_1) - 1}{L(h)}\right), \quad \text{it holds} \quad \mathbb{E}[e^{\alpha \mathcal{T}_1^+}] < \infty\,.$$

Of course $\lambda > 0$ and $\|h^+\|_1 - \log(\|h^+\|_1) - 1 > 0$, and so $\mathcal{T}_1^+$ admits an exponential moment.

Since τ_1^+ is the first time after $U_1^{+,1}$ such that there were no jump during a time $L(h)$. Thus $\tau_1^+ = \mathcal{T}_1^+$ and since $\tau_1 \le \tau_1^+$, part (i) of the proposition is proved.

In order to prove (ii) it is enough to show that the distribution of W_1^+ admits exponential moments. Recall that

$$W_1^+ = N^{h^+}([0, \tau_1^+])\,.$$

According to [8] (see proof of Theorem 3.2 and proof of Theorem 3.4 therein),

$$\lim_{t \to +\infty} \frac{1}{t} \ln \mathbb{E}\left(e^{\theta N^{h^+}([0,t])}\right) = \lambda(\mathbb{E}(e^{\theta \mathcal{S}}) - 1) := \mu(\theta) < +\infty$$

as soon as $\theta < \|h^+\|_1 - \log(\|h^+\|_1) - 1$. Here $\mathcal{S}$ is distributed according to (see (3) in [8])

$$\mathbb{P}(\mathcal{S} = k) = \frac{e^{-k\|h^+\|_1}(k\|h^+\|_1)^{k-1}}{k!}\,.$$

It is thus immediate that $\mu(\theta)$ goes to 0 as θ goes to 0.

For $\varepsilon > 0$ we may thus choose θ small enough such that

$$\alpha_0 - 2\varepsilon \ge \mu(2\theta) + \varepsilon\,.$$

For this θ, one can find t_θ such that for $t \ge t_\theta$,

$$\mathbb{E}\left(e^{2\theta N^{h^+}([0,t])}\right) \le e^{t(\mu(2\theta)+\varepsilon)}\,.$$

It follows

$$\begin{aligned}
\mathbb{E}(e^{\theta W_1^+}) &= \mathbb{E}\left(e^{\theta N^{h^+}([0,\tau_1^+])}\right) \\
&\leq \sum_{k=1}^{\infty} \mathbb{E}\left(e^{\theta N^{h^+}([0,k])}\,\mathbb{1}_{k-1\leq\tau_1^+<k}\right) \\
&\leq \sum_{k=1}^{\infty} \left(\beta_k\,\mathbb{E}\left(e^{2\theta N^{h^+}([0,k])}\right) + \frac{1}{\beta_k}\mathbb{P}(k-1\leq\tau_1^+)\right) \\
&\leq A(t_\theta) + \sum_{k=[t_\theta]+1}^{\infty} \left(\beta_k\, e^{k(\mu(2\theta)+\varepsilon)} + \frac{\mathbb{E}(e^{(\alpha_0-\varepsilon)\tau_1^+})}{\beta_k}\, e^{-(k-1)(\alpha_0-\varepsilon)}\right)
\end{aligned}$$

where $A(t_\theta)$ denotes the finite sum up to $k = [t_\theta]$. Choosing $\beta_k = k^{-2}\, e^{-k(\mu(2\theta)+\varepsilon)}$ the k'th term of the remaining sum is smaller than $1/k^2 + c\,k^2\, e^{-\varepsilon(k-1)}$ and the series is thus convergent. Since ε is arbitrary, (ii) follows. □

Proof of Proposition 3.2.7.

We consider a process N^h, generated by the Poisson point process Q, as in the Proposition 3.2.2. Since $h \leq 0$, we will couple (and upper-bound) this time the Hawkes process with the Poisson point process $\mathcal{R}$ on $\mathbb{R}^+$, with intensity λ, generated by the same Poisson point process Q on $(0,\infty)^2$. Since $\forall t \geq 0$

$$\lambda \geq \Lambda^h(t) \quad \text{a.s.}$$

we deduce that

$$\mathcal{R} \geq N^h.$$

We can now upper bound the length of the first time window τ_1 by a similar quantity associated with $\mathcal{R}$. Recall that U_1^1 is the first jump time of N^h and define:

$$\tau = \inf\{t > U_1^1, \mathcal{R}[t-L(h),t) \neq 0, \mathcal{R}(t-L(h),t] = 0\}. \tag{3.4.1}$$

τ indicates the first moment such that there were no jump for the process $\mathcal{R}$ during an interval of length $L(h)$. In particular, there weren't jump for N^h either. Therefore $\tau_1 \leq \tau$ a.s. and

$$W_1 = N^h([U_1^1,\tau_1]) \leq \mathcal{R}([U_1^1,\tau_1]) \leq \mathcal{R}([0,\tau]).$$

It is thus enough to get an upper bound for $\mathbb{E}\left(e^{\theta\,\mathcal{R}([0,\tau])}\right)$.

To this end we shall study the random variable τ. Denote by V_i the jumps of the Poisson point process $\mathcal{R}$. From the definition there exists a random integer K such that

$$\tau = V_K + L(h).$$

The definition of K leads to

$$K = \mathcal{R}[0,\tau].$$

From the independence of the times between jumps of $\mathcal{R}$ we deduce that

$$\begin{aligned}
\mathbb{P}(K=1) = \mathbb{P}[\tau = V_1 + L(h)] &= \mathbb{P}[V_2 - V_1 \geq L(h)] \\
&= \mathrm{e}^{-\lambda L(h)}, \\
\mathbb{P}(K=2) = \mathbb{P}[\tau = V_2 + L(h)] &= \mathbb{P}[\{V_2 - V_1 < L(h)\} \cup \{V_3 - V_2 \geq L(h)\}] \\
&= \mathbb{P}[V_2 - V_1 < L(h)]\mathbb{P}[V_3 - V_2 \geq L(h)] \\
&= (1 - \mathrm{e}^{-\lambda L(h)})\, \mathrm{e}^{-\lambda L(h)}, \\
\forall k \geq 2, \quad \mathbb{P}(K=k) = \mathbb{P}[\tau = V_k + L(h)] &= (1 - \mathrm{e}^{-\lambda L(h)})^{k-1}\, \mathrm{e}^{-\lambda L(h)}\,.
\end{aligned}$$

K is a geometric random variable with parameter $e^{-\lambda L(h)}$ and thus admits exponential moments provided $e^{\theta}\,(1 - e^{-\lambda L(h)}) < 1$ which concludes the proof. □

3.4.2 Proof of the LLN and CLT

Proof of Theorem 3.2.9 and Theorem 3.2.10.

Recall that

$$\hat{N}_t^h \leq N_t^h \leq \hat{N}_t^h + W_{M_t^h+1} \quad \text{a.s.}$$

where

$$\hat{N}_t^h := \sum_{i=1}^{\infty} W_i \mathbb{1}_{S_i \leq t} = \sum_{i=1}^{M_t^h} W_i$$

and

$$M_t^h := \sum_{i=1}^{\infty} \mathbb{1}_{S_i \leq t}\,,$$

as explained in (3.2.6), (3.2.7), (3.2.9).

As we previously said Theorem 3.1 and Theorem 3.2 Chapter 6 in [1] furnish

$$\frac{\hat{N}_t^h}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\mathbb{E}[W_1]}{\mathbb{E}(\tau_1)}$$

and

$$\sqrt{t}\left(\frac{\hat{N}_t^h}{t} - \frac{\mathbb{E}[W_1]}{\mathbb{E}(\tau_1)}\right) \underset{t\to\infty}{\Longrightarrow} \mathcal{N}(0,\sigma^2)$$

with

$$\sigma^2 = \frac{\operatorname{Var}\left(W_1 - \tau_1 \frac{\mathbb{E}[W_1]}{\mathbb{E}(\tau_1)}\right)}{\mathbb{E}(\tau_1)}\,.$$

It is thus enough to control the remaining (or error) term $W_{M_t^h+1}$ i.e to prove

$$\lim_{t\to\infty} \frac{W_{M_t^h+1}}{t} = 0 \quad \text{a.s.}, \tag{3.4.2}$$

and

$$\lim_{t\to\infty} \frac{W_{M_t^h+1}}{\sqrt{t}} = 0 \quad \text{in Probability.} \tag{3.4.3}$$

(3.4.2) will conclude the LLN and (3.4.3) the CLT.

Actually we will prove stronger results. Let $\beta(n)$ an increasing sequence going to infinity and $\varepsilon > 0$. Introduce the independent events $A_n = \{W_n > \varepsilon\beta(n)\}$. Then $\limsup_n A_n = \{\limsup_n \frac{W_n}{\beta(n)} > \varepsilon\}$. Since the $(W_i)_i$ are finite i.i.d random variables

$$\sum_n \mathbb{P}(A_n) = \sum_n \mathbb{P}\left(W_n > \varepsilon\beta(n)\right) = \sum_n \mathbb{P}(W_1 > \varepsilon\beta(n)) \,.$$

Thanks to Proposition 3.2.6 and to Markov inequality, we know that for

$$\mathbb{P}(W_1 > \varepsilon\beta(n)) \leq \mathbb{E}[e^{\theta_0 W_1}]\, e^{-\theta_0\,\varepsilon\,\beta(n)} \,.$$

We may now apply Borel-Cantelli, telling that provided $\sum_n e^{-\theta_0\,\varepsilon\,\beta(n)} < +\infty$,

$$\mathbb{P}(\limsup_n A_n) = 0.$$

The previous holds with $\beta(n) = n^\alpha$ for any $\alpha > 0$. We have proved in particular that

$$\frac{W_n}{\sqrt{n}} \xrightarrow[n\to\infty]{a.s.} 0 \,.$$

Since M_t^h is a non-decreasing family of integers going to infinity almost surely,

$$\frac{W_{M_t^h+1}}{\sqrt{M_t^h+1}} \xrightarrow[t\to\infty]{a.s.} 0 \,.$$

It remains to recall that

$$\frac{M_t^h+1}{t} \xrightarrow[t\to\infty]{} \frac{1}{\mathbb{E}(\tau_1)} \quad \text{a.s.,} \tag{3.4.4}$$

to conclude that

$$\frac{W_{M_t^h+1}}{\sqrt{t}} \xrightarrow[t\to\infty]{a.s.} 0 \,.$$

□

3.4.3 Proof of Corollary 3.2.13

Proof of Corollary 3.2.13.

In order to prove the first part of Corollary 3.2.13 it is enough to show that $\hat{N}_t^h/t$ and N_t^h/t are exponentially equivalent, i.e. that for each $\delta > 0$,

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\left| \frac{N_t^h}{t} - \frac{\hat{N}_t^h}{t} \right| > \delta \right) = -\infty \,.$$

To this end it is enough to show that for all $\delta > 0$

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(W_{M_t^h+1} > \delta t\right) = -\infty\,. \tag{3.4.5}$$

We will decompose the probability space into two events: $M_t^h \le t^2$ and $M_t^h > t^2$. It holds

$$\begin{aligned}
&\mathbb{P}\left(W_{M_t^h+1} > \delta\, t\right)\\
&\le \mathbb{P}\left(M_t^h > t^2\right) + \mathbb{P}\left(\left\{W_{M_t^h+1} > \delta t\right\} \cap \left\{M_t^h \le t^2\right\}\right)\\
&\le \mathbb{P}\left(M_t^h > t^2\right) + \mathbb{P}\left(\left\{\exists k \in \{1, ..., \lfloor t^2+1\rfloor\}, W_k > \delta t\right\} \cap \left\{M_t^h \le t^2+1\right\}\right)\\
&\le \mathbb{P}\left(M_t^h > t^2\right) + \mathbb{P}\left(\exists k \in \{1, ..., \lfloor t^2+1\rfloor\}, W_k > \delta t\right)\\
&\le \mathbb{P}\left(M_t^h > t^2\right) + \sum_{j=1}^{\lfloor t^2+1\rfloor} \mathbb{P}\left(W_k > \delta t\right)\\
&\le \mathbb{P}\left(M_t^h > t^2\right) + (t^2+1)\mathbb{P}\left(W_1 > \delta t\right).
\end{aligned}$$

On one hand, we have, by Markov's inequality, for all $\theta_0 > \theta > 0$,

$$\mathbb{P}\left(W_1 > \delta t\right) \le \mathbb{E}[\mathrm{e}^{\theta W_1}]\ \mathrm{e}^{-\theta\delta t}$$

so that

$$\limsup_{t\to+\infty} \frac{1}{t} \ln(1+t^2)\,\mathbb{P}\left(W_1 > \delta t\right) \le -\theta_0\delta.$$

On the other hand, according to [80] Theorem 2.3, for all $x > 0$:

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{M_t^h}{t} \ge x\right) \le -J_{\tau_1}(x),$$

where $J_{\tau_1}(x) = \sup_\eta\{\eta - x\ln \mathbb{E}[\mathrm{e}^{\eta\tau_1}]\}$.
Since $\eta \mapsto \mathbb{E}(e^{\eta\tau_1})$ is continuous on $\mathbb{R}^-$ there exists some η_0 such that $\mathbb{E}(e^{\eta_0\tau_1}) = e^{-1}$.
It follows $J_{\tau_1}(x) \ge \eta_0 + x$.
Choose $t_1, t_2, ...$ an increasing sequence of times such that $t_i \underset{i\to+\infty}{\longrightarrow} +\infty$. For a fixed i, we have for t large enough

$$\mathbb{P}\left(M_t^h > t^2\right) = \mathbb{P}\left(\frac{M_t^h}{t} > t\right) \le \mathbb{P}\left(\frac{M_t^h}{t} > t_i\right)$$

Since

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{M_t^h}{t} > t_i\right) \le -J_{\tau_1}(t_i) \le -t_i - \eta_0.$$

It follows,

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{M_t^h}{t} > t\right) = -\infty.$$

Eventually,

$$\begin{aligned}
&\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{W_{M_t^h+1}}{t} > \delta\right) \\
&\quad \leq \limsup_{t\to+\infty} \frac{1}{t} \ln \left[\mathbb{P}\left(M_t^h > t^2\right) + (t^2+1)\mathbb{P}\left(W_1 > \delta t\right)\right] \\
&\quad \leq \limsup_{t\to+\infty} \left(\frac{\ln 2}{t} + \max\left[\frac{1}{t}\ln \mathbb{P}\left(\frac{M_t^h}{t} > t\right), \frac{1}{t}\ln\left((t^2+1)\mathbb{P}\left(W_1 > \delta t\right)\right)\right]\right) \\
&\quad \leq \max\left[\limsup_{t\to+\infty} \frac{1}{t}\ln \mathbb{P}\left(\frac{M_t^h}{t} > t\right), \limsup_{t\to+\infty}\left(\frac{\ln(t^2+1)}{t} + \frac{1}{t}\ln \mathbb{P}\left(W_1 > \delta t\right)\right)\right] \\
&\quad \leq -\theta_0\, \delta\,.
\end{aligned}$$

This completes the proof for $\theta_0 = +\infty$.

Let us now assume $\theta_0 < \infty$. Recall that $m = \frac{\mathbb{E}(W_1)}{\mathbb{E}(\tau_1)}$, then (3.2.10) is a consequence of the following line of reasoning:

$$\begin{aligned}
\mathbb{P}\left(\frac{N_t^h}{t} \geq m+a\right) &\leq \mathbb{P}\left(\frac{\hat{N}_t^h}{t} + \frac{W_{M_t^h+1}}{t} \geq m+a\right) \\
&\leq \mathbb{P}\left(\frac{\hat{N}_t^h}{t} \geq m+\kappa_1 a\right) + \mathbb{P}\left(\frac{W_{M_t^h+1}}{t} \geq (1-\kappa_1)a\right)
\end{aligned}$$

where $\kappa_1 \in (0,1)$, yielding

$$\begin{aligned}
&\limsup_{t} \frac{1}{t} \ln \mathbb{P}\left(\frac{N_t^h}{t} \geq m+a\right) \\
&\quad \leq \max\left(\limsup_{t} \frac{1}{t}\ln \mathbb{P}\left(\frac{\hat{N}_t^h}{t} \geq m+\kappa_1 a\right), \limsup_{t} \frac{1}{t}\ln \mathbb{P}\left(\frac{W_{M_t^h+1}}{t} \geq (1-\kappa_1)a\right)\right)
\end{aligned}$$

Now applying Theorem 3.2.12 with κ_2 and $(1-\kappa_2)$, we deduce that

$$\begin{aligned}
&\limsup_{t} \frac{1}{t} \ln \mathbb{P}\left(\frac{N_t^h}{t} \geq m+a\right) \\
&\quad \leq \max\left(-\inf_{z-m\geq \kappa_2\kappa_1 a} J(z)\,, -(1-\kappa_2)\kappa_1\theta_0 a\,, -(1-\kappa_1)a\theta_0\right)
\end{aligned}$$

yielding the result with $\kappa = \kappa_1\kappa_2$ and $\kappa' = 1-\kappa_1$. The condition $\kappa + 2\kappa' = 1$ arises from the equality of the last two terms.

Finally, (3.2.11) is a consequence of the same reasoning on $\hat{N}_t^h \leq N_t^h$:

$$\mathbb{P}\left(\frac{N_t^h}{t} \leq m-a\right) \leq \mathbb{P}\left(\frac{\hat{N}_t^h}{t} \leq m-a\right)$$

yielding the result. □

3.5 Final comments.

As we recalled in the introduction, in the linear case the LLN, the CLT and the LDP are completely characterized by $||h||_1$. As we have shown in Section 3.3, the "almost linear case with inhibition" we are looking at is dramatically different, since the limiting behaviour is not fully determined by $||h||_1$ nor even by some moments of h. The renewal description of the Hawkes process we have used allows us to characterize all these limit theorems in terms of the joint law of (τ_1, W_1). It should be very interesting to link this distribution with h. As for the non linear self-excited case such a goal seems difficult to reach.

Another interesting direction should be to obtain non asymptotic deviation bounds (or concentration properties). Since the Large Deviation Principle for cumulative processes we have proved in [20] is based on the contraction of a higher level LDP, new methods are necessary for non asymptotic results.

The methods of the paper can be used for more general jump rate functions f, provided one can generalize the construction of the sequence (τ_i, W_i). This generalization is partly done in [38] in which a regenerative structure is exhibited without the assumption of bounded support for the reproduction function h and in [69] which exhibit renewal points for non linear Hawkes processes and age-dependent Hawkes processes.

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Chapter 4

Uniform in time propagation of chaos in networks of FitzHugh-Nagumo neurons

This chapter is a joint work with Pierre Le Bris. It will soon be submitted.

Abstract

In this article, we are interested in the behavior of a fully connected network of N neurons, where N tends to infinity. We assume that neurons follow the stochastic FitzHugh-Nagumo model, whose specificity is the non-linearity with a cubic term. We prove a result of uniform in time propagation of chaos of this model in a mean-field framework. We also exhibit explicit bounds. We use a coupling method initially suggested by A. Eberle [31], and recently extended by [29].

Contents

4.1 Introduction

4.1.1 Understanding the model

Understanding the brain activity is both a complex and important challenge in current research. Of course, interests are plentiful: characterizing brain functions, understanding structures and links between them and figuring out some phenomena - such as cyclic heartbeat. A way of modeling this activity is by considering a very large number of individual neurons with interactions. Since the number of neurons in a human brain is around 10^{11}, and even "small" parts of the brain are thus constituted of very large number of them, such a strategy can be considered coherent.

The main quantity we study is the membrane potential of the nerve cells: it can "easily" be observed and its modification characterizes a synapse (an interaction between neurons). Neurons regulate their electrical potential. In general, without interaction, the potential evolves with time but has quite small changes. Incoming potentials from other neurons are usually what make the neuron fire, i.e. send potential to other neurons. We will here focus on an homogeneous network of neurons and consider mean-field interactions. This way, each neuron will interact with every other one, as it can be the case in small regions of the brain. The parameters of the model will be considered the same for each neuron.

A classical model was introduced by Hodgkin and Huxley [45] using experimental data of the activity of the giant squid axon. It describes the ion exchanges K^+, Na^+ and Cl^- through the membrane and their effects on the potential. A simplification of this model is the FitzHugh-Nagumo model, which reduces the dimension: from four-dimensional model (for one neuron) with Hodgkin-Huxley equations, we obtain a two-dimensional model. It's a compromise between the biological accuracy and the mathematical simplicity.

The deterministic FitzHugh-Nagumo model for one neuron (or one particle) is given by the following equations:

$$\begin{cases} dX_t = (X_t - (X_t)^3 - C_t - \alpha)dt \\ dC_t = (\gamma X_t - C_t + \beta)dt, \end{cases}$$

where X is the potential membrane and C is a recovery variable, called the adaptation variable. The parameters γ and β are positive constants that determine the duration of an excitation and the position of the equilibrium point of this system. Finally $\alpha \in \mathbb{R}$ is the magnitude of a stimulus current (an entrance current in the system). Note that the variable C isn't a physical quantity, and is used to allow X to mimic the behavior of the potential. This variable C has linear dynamics and provides a slower negative feedback.

Stochastic versions of the model, adding some noise on one or both coordinates have also been introduced. Noise models different types of randomness depending on the coordinates: when the noise is in the first equation (dynamics of X) with a standard deviation $\sigma_X > 0$, it models a noisy presynaptic current. When it is in the second equation (dynamics of C) with a standard deviation $\sigma_C > 0$, it describes a noisy conductance dynamic (a noise in the chemical behavior). In general, noise in this model is additive. Various mathematical questions can be studied. Some authors choose to focus on the properties of the natural macroscopic limit of the model as $N \to \infty$ when it is clearly defined (see system (4.1.2)), when others work on properties of the particles system for fixed N. These models can be quite complicated to study mathematically. The main objectives are to characterize the behavior of these models when the number of neurons N tends to $+\infty$ in a mean-field limit, and to prove whether or not there exists an equilibrium, a stationary behavior, when t tends to $+\infty$. The question of the synchronization of neurons can also be studied, since it is a phenomenon observed in different contexts, such as the generation of respiratory rhythm or complex neurological functionalities.

Some authors study a model with two noises: in this case, the stochastic system is non-degenerate and is then elliptic.
Berglund and Landon describe the behavior of the deterministic FitzHugh-Nagumo model for one neuron in [6], and consider the stochastic model to work on the behavior of the interspike interval and the distribution of oscillations of the solution. In [78], Tatchim Bemmo, Siewe Siewe and Tchawoua focus on a quite different stochastic model by considering additive noise η on the dynamics of X, and multiplicative noise ξ on the dynamics of C, both defined as sinusoidal function of correlated Brownian motions. They choose to avoid Gaussian noises since it is an unbounded noise. They also consider a deterministic and periodic entrance signal in the first equation. They observe abrupt transitions of the potential membrane X when the intensity of the noise is gradually changed.

In general, a lot of authors focus on a noise on only one variable, thus the stochastic model is degenerate. In [81], the authors work on the determination of firing time. They consider a stochastic FitzHugh-Nagumo model for one neuron, with Brownian noise on X, obtain approximation of firing times and compare them with numerical simulations. Thieullen describes the behavior of the solution of one deterministic FitzHugh-Nagumo system and extends the result in the case of a stochastic FitzHugh-Nagumo system, in [79]. In her case, the stochasticity is a Brownian noise on the dynamics of X. In [53], León and Samson consider a FitzHugh-Nagumo model with a noise on C but not on X, i.e. $\sigma_X = 0$, and study the properties of the equations for one neuron. In particular, they focus on hypoellipticity of the model, the existence and uniqueness of an invariant probability and a mixing property by establishing a link between the model and the class of stochastic damping Hamiltonian systems. They also consider neuronal modeling questions and study the generation of spikes in function of parameters of the model. On the contrary, the article [82] focuses on stochastic FitzHugh-Nagumo model with noise in the dynamics of X, and $\sigma_C = 0$. They study one neuron in a periodically forced regime. This study relies on the theory of Markovian Random Dynamical System. The model is driven by a cosinus signal, and Uda studies the spike rate and compares it with the probability of two-points motion of membrane potential.

Here we consider mean-field interactions. These interactions are described by two functions K_X and K_C, applied on the difference between two states $((X_t^i, C_t^i) - (X_t^j, C_t^j))$, as described in (4.1.1). In particular, this type of interaction models electrical synapses, which are a sort of synapses.

In [4], Baladron, Fasoli, Faugeras and Touboul study FitzHugh-Nagumo and Hodgkin-Huxley models with mean-field interaction, only on X (i.e. $K_C = 0$). They consider more general interactions, not only applied on the difference between two states, modeling chemical synapses and electrical synapses. For the FitzHugh-Nagumo model, they consider a noise on X. They prove propagation of chaos, i.e. the convergence of the law of k neurons towards the law of k independent solutions of the mean-field equations. This article is completed and clarified by the work of Bossy, Faugeras and Talay in [12].
Mischler, Quininao and Touboul consider a FitzHugh-Nagumo model in [63], with a linear interaction on X, and a noise only on X, i.e. $\sigma_C = 0$ and $K_X(x, c) = \lambda x$. The drift on X is not exactly the same as in the model above, but remains a cubic function of X, and is very similar. They work on the properties of a solution of the McKean-Vlasov evolution PDE associated to this model and obtain the uniqueness of a global weak solution. Furthermore, they prove that there exists at least one stationary solution, and when the interaction is small, the stationary solution is unique and exponentially stable. They also exhibit numerical results with open problems, like attractive periodic solution in time. [18] completes their work on the properties of the solution. In a closed framework, in [59], Luçon and Poquet study the macroscopic limit of a mean-field model, where they introduce a slow-fast analysis by considering a slow deterministic part, and a fast stochastic and interaction part. In particular, they focus on the periodicity of such a system. They analyze the influence of both noise and interaction on the emergence of periodic behavior, and prove the existence of periodic solution, exponentially attractive, when the parameters satisfy some assumptions and the drift is "small" enough with respect to interaction and noise.

This model can be made more complex, by considering non mean-field interaction. In particular, Bayrak, Hövel and Vuksanović work on a stochastic FitzHugh-Nagumo model with a network interaction in [5]. Their type of interaction take into account a connectivity coefficient between two neurons, and a propagation velocity.

Other authors choose to consider stochastic FitzHugh-Nagumo with a spatial model. A second spatial derivative of X is added in the dynamics of X. [61], [54], [56] and [55] explore the notion of random attractors in these models. Numerical schemes for the interacting particles system are also studied, for instance in [71].

4.1.2 Framework and results

Combining noise and interaction, we work specifically on the following equations, for $1 \leq i \leq N$, where N is the number of neurons:

$$\begin{cases} dX_t^{i,N} = (X_t^{i,N} - (X_t^{i,N})^3 - C_t^{i,N} - \alpha)dt + \frac{1}{N}\sum_{j=1}^N K_X(Z_t^{i,N} - Z_t^{j,N}) + \sigma_X dB_t^{i,X} \\ dC_t^{i,N} = (\gamma X_t^{i,N} - C_t^{i,N} + \beta)dt + \frac{1}{N}\sum_{j=1}^N K_C(Z_t^{i,N} - Z_t^{j,N}) + \sigma_C dB_t^{i,C}, \end{cases} \tag{4.1.1}$$

where we denote by Z_t^i the couple (X_t^i, C_t^i) to simplify the notations.
We assume $(B_t^{i,X})_i$ and $(B_t^{i,C})$ to be independent Brownian motions. Here, we consider two Brownian noises B^X and B^C, one on each equation, and thus assume that each neuron has its own independent noises, and that there is no environmental (or shared) noise.
We also assume K_X and K_C to be Lipschitz continuous and respectively denote their Lipschitz constants by L_X and L_C.

The goal of this article is to describe the behavior of this network as the number N of neurons tends to infinity.

To describe its behavior, we consider the $\mathbb{R}^2$-valued process $(\bar{Z}_t)_{t\geq 0} = (\bar{X}_t, \bar{C}_t)_{t\geq 0}$ evolving according to the following non-linear stochastic differential equation of *McKean-Vlasov* type

$$\begin{cases} d\bar{X}_t = (\bar{X}_t - (\bar{X}_t)^3 - \bar{C}_t - \alpha)dt + K_X * \bar{\mu}_t(\bar{Z}_t)dt + \sigma_x d\bar{B}_t^X \\ d\bar{C}_t = (\gamma \bar{X}_t - \bar{C}_t + \beta)dt + K_C * \bar{\mu}_t(\bar{Z}_t)dt + \sigma_c d\bar{B}_t^C, \end{cases} \tag{4.1.2}$$

where $\bar{\mu}_t = \text{Law}(\bar{Z}_t)$ is the law at time t of the process $(\bar{X}_t, \bar{C}_t)$, and $*$ denotes the operation of convolution, i.e.

$$K * \nu(u) = \int K(u-v)\nu(dv).$$

To some extent, (4.1.1) can be seen as an approximation of (4.1.2) in which the operation of convolution is applied to the empirical measure $\mu_{t,\text{emp}} = \frac{1}{N}\sum_{i=1}^N \delta_{Z_t^i}$. We wish to prove that the law μ_t^N of the network (4.1.1) converges in some sense to $\bar{\mu}_t^{\otimes N}$ (i.e the law of the solution of (4.1.2) tensorized N times) as N tends to infinity. This phenomenon has been stated under the name *propagation of chaos* -an idea motivated by M. Kac [50]) and greatly developed by A.S. Sznitman [77]- as it amounts to saying that, as the number of particles increases in the system, two particles will become "more and more" independent, converging towards a tensorized law. The notion of "propagation" refers to the fact that proving such convergence at time 0 is sufficient to prove it at a later time t.

In order to prove the convergence of μ_t^N to $\bar{\mu}_t^{\otimes N}$, we follow the coupling method described in a recent work by one of the authors in [40], the result of which cannot be applied directly here. This method has been put forward by Eberle, following earlier works by Lindvall and Rogers [57]. Let us briefly recall the method.

We consider $r_t^i = |\bar{X}_t^i - X_t^{i,N}| + \delta|\bar{C}_t^i - C_t^{i,N}|$ with $\delta > 0$, a constant not yet specified.

A natural distance between probability measures is the Wasserstein distance, linked to the theory of optimal transport. For μ and ν two probability measures on $\mathbb{R}^d$, we denote

$$\mathcal{W}_p(\mu,\nu) = \inf_{X\sim\mu,\ Y\sim\nu} \mathbb{E}\left(||X-Y||_p^p\right)^{1/p}, \tag{4.1.3}$$

where $||\cdot||_p$ denotes the usual L^p distance on $\mathbb{R}^d$. It is thus defined as the minimum over all possible choices of a pair (X,Y), such that X is distributed according to μ and Y according to ν, of the expectation of the distance between X and Y. The basic idea behind a coupling method is then that an upper bound on the Wasserstein distance between μ and ν is given by the construction of any pair of random variables distributed according to these probability measures. Thus, instead of considering the minimum over all possible coupling, we construct simultaneously two solutions of (4.1.1) and (4.1.2) that will tend to get closer together as the number of neurons increases.

Have $\left(\bar{X}_t^i, \bar{C}_t^i\right)$, for i between 1 and N, be N independent copies of a solution of (4.1.2) driven by some independent Brownian motions $(\bar{B}_t^{i,X})_{t\geqslant 0}$ and $(\bar{B}_t^{i,C})_{t\geqslant 0}$. A coupling of $\left(\bar{X}_t^i, \bar{C}_t^i\right)$ and $\left(X_t^{i,N}, C_t^{i,N}\right)$ then follows from a coupling of the Brownian motions B and $\bar{B}$.

The first natural choice, popularized by Sznitman [77], is the *synchronous* coupling and consists in choosing $B = \bar{B}$. By doing so, when considering the time evolution of $\bar{Z}_t^i - Z_t^{i,N} = \left(\bar{X}_t^i - X_t^{i,N}, \bar{C}_t^i - C_t^{i,N}\right)$, the noise cancels out. The contraction of a distance between the processes can then only be induced by the deterministic drift, as in [7], and this usually only holds under rather restrictive conditions (in particular the drift should be strongly convex). Nevertheless, in our case, the calculation of the evolution of $\bar{X}_t^i - X_t^{i,N}$ and $\bar{C}_t^i - C_t^{i,N}$ (see later) shows that there is still some deterministic contraction when $\bar{X}_t^i - X_t^{i,N} = 0$. We can therefore use a synchronous coupling in the vicinity of this subspace.

Outside of this subspace, we use the noise to get the processes closer together. In the direction orthogonal to this space we consider $B = -\bar{B}$, as this maximizes the variance of the noise. This yields the use of the *reflection* coupling of [57]. Notice however at this stage that, because of the symmetry of the noise, there is *a priori* no reason r_t^i should decrease rather than increase. This invites us to consider $f(r_t^i)$, with f a concave function, so that a random decrease has more effect than a random increase of the same value. We will define the function f later.

Finally we construct a Lyapunov function H to take into account the trend of each process to come back to some compact set of $\mathbb{R}^2$. We are then led to the study of a suitable distance between the two processes, which will be of the form $\rho_t := \frac{1}{N}\sum_{i=1}^N f(r_t^i)(1+\epsilon H(\bar{Z}_t^i)+\epsilon H(Z_t^{i,N}))$, where $\epsilon > 0$. This quantity controls the usual L^1 and L^2 distances between the two systems, and is interesting as, when r_t^i is small, $f(r_t^i)$ tends to decrease either because of the deterministic drift or the reflection coupling, and when r_t^i is big, the Lyapunov functions H will tend to decrease. We thus show that $\mathbb{E}\rho_t$ decays exponentially fast. This leads to several constraints on δ, ϵ and on the parameters involved in the definition of f, and we have to prove that it is possible to meet all these conditions simultaneously. In reality, the quantity ρ_t considered will be a slight twist of the one given above (see (4.2.26)) so as to take into account the non linearity of the process.

As explained, this method requires some noise in the direction orthogonal to the naturally contracting subspace. This means, in the description of the method above, that one should have $\sigma_X > 0$ (so that we can use a reflection coupling to bring $\bar{X}_t^i$ and $X_t^{i,N}$ closer together). In the case $\sigma_X = 0$ and $\sigma_C > 0$, a modification of the calculations is necessary. We describe this case in a separate section.

Assumption 4.1.1. Let $L_{X,max}$ and $L_{C,max}$ be two universal constants

$$\exists L_X \in [0, L_{X,max}], \forall z, z' \in \mathbb{R}^2 \quad |K_X(z) - K_X(z')| \leq L_X(\|z - z'\|_1)$$

$$\exists L_C \in [0, L_{C,max}], \forall z, z' \in \mathbb{R}^2 \quad |K_C(z) - K_C(z')| \leq L_C(\|z - z'\|_1)$$

$$K_X(0,0) = 0 \text{ and } K_C(0,0) = 0$$

These constants $L_{X,max}$ and $L_{C,max}$ might seem at first glance off putting as they are not given. When we prove non uniform in time propagation of chaos, these constants can be chosen to be $+\infty$. When we prove uniform in time propagation of chaos however, $L_{X,max}$ and $L_{C,max}$ are *a priori* bounds : Theorem 4.1.4 below will be true for L_X and L_C sufficiently small. The condition $L_X \leq L_{X,max}$ and $L_C \leq L_{C,max}$ are therefore not restrictive conditions, and are useful in proving some parameters are independent of L_X and L_C. Remark 4.2.7 below shows that one can for instance consider $L_{X,max} = 4$ and $L_{C,max} = \frac{1}{5}$.

Before any result on propagation of chaos, we prove that both systems (4.1.1) and (4.1.2) have well-defined solutions:

Proposition 4.1.2. Let K_X and K_C satisfy Assumptions 4.1.1, with $L_{X,max} = \infty$ and $L_{C,max} = \infty$.
There exists a unique and strong solution for system (4.1.1) and a unique and strong solution for system (4.1.2).

Now, we can state our results on propagation of chaos. We denote $\mathcal{W}_1$ and $\mathcal{W}_2$ the usual L^1 and L^2 Wasserstein distances.

Theorem 4.1.3. [Non uniform in time propagation of chaos] Let K_X and K_C satisfy Assumptions 4.1.1, with $L_{X,max} = \infty$ and $L_{C,max} = \infty$. There exist explicit

$C_1, C_2 > 0$, such that for all probability measures μ_0 on $\mathbb{R}^2$ with finite second moment,

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \le C_1 e^{C_2 t} \frac{k}{\sqrt{N}},$$

for all $k \in \mathbb{N}$, where $\mu_t^{k,N}$ is the marginal distribution at time t of the first k neurons $\left((X_t^1, C_t^1), ..., (X_t^k, C_t^k)\right)$ of an N particles system (4.1.1) with initial distribution $(\mu_0)^{\otimes N}$, while $\bar{\mu}_t$ is a solution of (4.1.2) with initial distribution μ_0.

Our main result consist in removing the time dependency in the previous upperbound. This uniform propagation of chaos requires stronger assumptions on the interaction kernels.

Theorem 4.1.4. [Uniform in time propagation of chaos] Let $\tilde{\mathcal{C}}^0 > 0$ and $\tilde{a} > 0$. There is an explicit $\mathcal{C}_{K_X,K_C} > 0$ such that, for all K_X and K_C satisfying Assumptions 4.1.1 with $L_X, L_C < \mathcal{C}_{K_X,K_C}$, there exist explicit $B_1, B_2 > 0$, such that for all probability measures μ_0 on $\mathbb{R}^2$ satisfying $\mathbb{E}_{\mu_0}\left(e^{\tilde{a}(|X|+|C|)}\right) \le \tilde{\mathcal{C}}^0$,

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \le B_1 \frac{k}{\sqrt{N}}, \qquad \mathcal{W}_2^2\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \le B_2 \frac{k}{\sqrt{N}},$$

for all $k \in \mathbb{N}$, where $\mu_t^{k,N}$ is the marginal distribution at time t of the first k neurons $\left((X_t^1, C_t^1), ..., (X_t^k, C_t^k)\right)$ of an N particles system (4.1.1) with initial distribution $(\mu_0)^{\otimes N}$, while $\bar{\mu}_t$ is a solution of (4.1.2) with initial distribution μ_0.

$\mathcal{C}_{K_X,K_C}$ is explicitly given, depending on other parameters in 4.2.24. It's a non optimal definition.

As previously mentioned, this work follows the method described in [40]. Beyond the result of uniform in time propagation of chaos for the FitzHugh-Nagumo model, which is in itself an interesting result, the present work is also a testament to the robustness of the coupling method.

4.1.3 Existence of solutions

First of all, we prove Proposition 4.1.2, i.e existence of strong solutions of systems (4.1.1) and (4.1.2), under Assumption 4.1.1, and we begin with (4.1.1).
Let's denote, for $K \in \mathbb{R}^+$,

$$g_K(x) = \begin{cases} -K^3 & \text{if } x < -K \\ x^3 & \text{if } x \in [-K, K] \\ K^3 & \text{if } x > K. \end{cases}$$

g_K is locally Lipschitz and is bounded.

Thus, it's well known (see Chapter 3 [47]) that the following system (under Assumption 4.1.1)

$$\begin{cases} dX_t^{i,N} = (X_t^{i,N} - g_K(X_t^{i,N}) - C_t^{i,N} - \alpha)dt + \frac{1}{N}\sum_{j=1}^N K_X(Z_t^{i,N} - Z_t^{j,N}) + \sigma_X dB_t^{i,X} \\ dC_t^{i,N} = (\gamma X_t^{i,N} - C_t^{i,N} + \beta)dt + \frac{1}{N}\sum_{j=1}^N K_C(Z_t^{i,N} - Z_t^{j,N}) + \sigma_C dB_t^{i,C}, \end{cases} \tag{4.1.4}$$

for $1 \leq i \leq N$, have strong and unique solution.

In consequence, for a fixed $K \in \mathbb{R}^+$, there exists a strong solution of system (4.1.1) until time $T_K = \sup\{t, \forall i, \forall s \leq t, X_t^i \leq K \text{ and } C_t^i \leq K\}$, and the solution coincide with the solution of the system with g_K.

We have the following Proposition:

Proposition 4.1.5. If, for each $i \leq N$, $\mathbb{E}(|X_0^{i,N}|^2) < +\infty$ and $\mathbb{E}(|C_0^{i,N}|^2) < +\infty$, then there exists $\mathcal{C}_t < \infty$ such that, for each $i \leq N$:

$$\mathbb{E}\left(|X_t^{i,N}|^2 + |C_t^{i,N}|^2\right) \leq \mathcal{C}_t. \tag{4.1.5}$$

The proof relies on the Lyapunov function defined in the next Section, and is given in Appendix 4.A.2.

Then, by denoting T_∞ the explosion time of a solution of system (4.1.1)

$$T_\infty = \inf\{t, \exists i, \forall A > 0, \exists \varepsilon > 0, \forall s \in (t-\varepsilon, t), X_s^{i,N} > A \text{ or } C_s^{i,N} > A\}$$

we deduce $\forall t \in \mathbb{R}^+, \mathbb{P}(T_\infty \leq t) = 0$. Eventually, there exists unique and strong solution for system (4.1.1).

The existence and unicity of a solution of (4.1.2) is known from the Theorem 3.3 from [70]. We only have to prove that the Assumptions 3.2 [70] are verified. We define, for all $t \in \mathbb{R}^+$, $z = (x, c) \in \mathbb{R}^2$ and for all probability distribution ν with a finite variance:

$$\begin{aligned} b(t, z, \nu) &= \begin{pmatrix} x - x^3 - c - \alpha + K_X * \nu(z) \\ \gamma x - c + \beta + K_C * \nu(z) \end{pmatrix} \\ \sigma(t, z, \nu) &= \begin{pmatrix} \sigma_X \\ \sigma_C \end{pmatrix}. \end{aligned}$$

σ is a constant function, so it clearly satisfies the different conditions.

For $t \in \mathbb{R}^+$, z, z' in $\mathbb{R}^2$, and ν a probability measure:

$$\begin{aligned} \langle z - z', & b(t, z, \nu) - b(t, z', \nu)\rangle \\ &= (x - x')\left((x - x') - (x^3 - x'^3) - (c - c') + K_X * \nu(z) - K_X * \nu(z')\right) \\ &\quad + (c - c')\left(\gamma(x - x') - (c - c') + K_C * \nu(z) - K_C * \nu(z')\right) \\ &= (x - x')^2 - (x - x')^2(x^2 + xx' + x'^2) + (\gamma - 1)(c - c')(x - x') - (c - c')^2 \\ &\quad + (K_X * \nu(z) - K_X * \nu(z')) + (K_C * \nu(z) - K_C * \nu(z')). \end{aligned}$$

Since $(x^2 + xx' + x'^2) \geq 0$, the second term is non-positive. K_X and K_C are Lipschitz function, so the last line is clearly bounded by $\|z - z'\|_2^2$ up to a multiplicative constant. Then, there exists a constant L such that

$$\langle z - z', b(t, z, \nu) - b(t, z', \nu)\rangle \leq L\|z - z'\|_2^2.$$

Since K_X and K_C are Lipschitz function, we also have, for all probability distribution ν and ν' with a finite variance,

$$\|b(t,z,\nu)-b(t,z,\nu')\|_2 \leq L\mathcal{W}_2(\nu,\nu').$$

Eventually, since b is Locally Lipschitz with polynomial growth, each Assumption is satisfied and Theorem 3.3 [70] can be applied. Note that we could also apply Proposition 2.19 from [60] : Assumptions are the same, and it gives a result for interaction depending on a spatial position.

To complete the Proposition 4.1.5, we also give the following

Proposition 4.1.6. If $\mathbb{E}(|\bar{X}_0|^2) < +\infty$ and $\mathbb{E}(|\bar{C}_0|^2) < +\infty$, then there exists $C_{0,1}$ and $C_{0,2}$ such that:

$$\mathbb{E}\left(|\bar{X}_t|^2 + |\bar{C}_t|^2\right) \leq C_{0,1}e^{C_{0,2}t}. \tag{4.1.6}$$

The proof is very similar with Proposition 4.1.5 and is in Appendix 4.A.2.

4.1.4 Quick result : non uniform in time propagation of chaos

We start by proving Theorem 4.1.3, a non uniform in time propagation of chaos, as it highlights the basic strategy behind a coupling argument. Some of the following expressions will be used in the proof of Theorem 4.1.4, in Section 4.3

We consider a synchronous coupling between $(Z_t^{i,N})_i$ and $(\bar{Z}_t^i)_i$, i.e. $\bar{B}_t^{i,X} = B_t^{i,X}$ and $\bar{B}_t^{i,C} = B_t^{i,C}$. We have

$$\begin{cases} dX_t^{i,N} = (X_t^{i,N} - (X_t^{i,N})^3 - C_t^{i,N} - \alpha)dt + \frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j)dt + \sigma_x dB_t^{i,X} \\ dC_t^{i,N} = (\gamma X_t^{i,N} - C_t^{i,N} + \beta)dt + \frac{1}{N}\sum_{j=1}^N K_C(Z_t^i - Z_t^j)dt + \sigma_c dB_t^{i,C} \end{cases}$$

and

$$\begin{cases} d\bar{X}_t^i = (\bar{X}_t^i - (\bar{X}_t^i)^3 - \bar{C}_t^i - \alpha)dt + K_X * \bar{\mu}_t(\bar{Z}_t^i)dt + \sigma_x dB_t^{i,X} \\ d\bar{C}_t^i = (\gamma\bar{X}_t^i - \bar{C}_t^i + \beta)dt + K_C * \bar{\mu}_t(\bar{Z}_t^i)dt + \sigma_c dB_t^{i,C}, \end{cases}$$

with $\bar{\mu}_t$ the law of $\bar{Z}_t^1$. The method is the following :

- we compute the time evolution of $\mathbb{E}(r_t^i)$ using Ito's formula,
- we control the difference between the drifts $\frac{1}{N}\sum_{j\neq i} K(\bar{Z}_t^i - \bar{Z}_t^j)$ and $K * \bar{\mu}_t(\bar{Z}_t^i)$ using some form of law of large number. This is where the convergence rate $\sqrt{N}$ appears,
- and we conclude using Gronwall's lemma.

Time evolution : We have,

$$\begin{aligned} d(X_t^{i,N} - \bar{X}_t^i) = \Bigg[&(X_t^{i,N} - \bar{X}_t^i) - \left((X_t^{i,N})^3 - (\bar{X}_t^i)^3\right) - (C_t^{i,N} - \bar{C}_t^i) \\ &+ \frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\Bigg] dt. \end{aligned}$$

We denote

$$\operatorname{sign}(x) = \begin{cases} \frac{x}{|x|} \text{ if } x \neq 0, \\ 0 \text{ otherwise,} \end{cases}$$

and obtain, using Ito's formula,

$$\begin{aligned} d|X_t^{i,N} - \bar{X}_t^i| = \Big[& \operatorname{sign}(X_t^{i,N} - \bar{X}_t^i)(X_t^{i,N} - \bar{X}_t^i) \\ & - \operatorname{sign}(X_t^{i,N} - \bar{X}_t^i)\left((X_t^{i,N})^3 - (\bar{X}_t^i)^3\right) \\ & - \operatorname{sign}(X_t^{i,N} - \bar{X}_t^i)(C_t^{i,N} - \bar{C}_t^i) \\ & + \operatorname{sign}(X_t^{i,N} - \bar{X}_t^i)\frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\Big] dt \\ \leq \Big[& |X_t^{i,N} - \bar{X}_t^i| - \left|(X_t^{i,N})^3 - (\bar{X}_t^i)^3\right| + \left|C_t^{i,N} - \bar{C}_t^i\right| \\ & + \left|\frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|\Big] dt. \end{aligned} \tag{4.1.7}$$

Similarly,

$$\begin{aligned} d(C_t^{i,N} - \bar{C}_t^i) = \Big[& \gamma(X_t^{i,N} - \bar{X}_t^i) - (C_t^{i,N} - \bar{C}_t^i) \\ & + \frac{1}{N}\sum_{j=1}^N K_C(Z_t^i - Z_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\Big] dt, \end{aligned}$$

and we obtain

$$\begin{aligned} d|C_t^{i,N} - \bar{C}_t^i| \leq \Big[& \gamma\left|X_t^{i,N} - \bar{X}_t^i\right| - |C_t^{i,N} - \bar{C}_t^i| \\ & + \left|\frac{1}{N}\sum_{j=1}^N K_C(Z_t^i - Z_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\right|\Big] dt. \end{aligned} \tag{4.1.8}$$

Thus, denoting $r_t^i = |X_t^{i,N} - \bar{X}_t^i| + |C_t^{i,N} - \bar{C}_t^i|$

$$\begin{aligned} dr_t^i \leq \Big[& (1+\gamma)|X_t^{i,N} - \bar{X}_t^i| - \left|(X_t^{i,N})^3 - (\bar{X}_t^i)^3\right| \\ & + \left|\frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right| \\ & + \left|\frac{1}{N}\sum_{j=1}^N K_C(Z_t^i - Z_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\right|\Big] dt. \end{aligned}$$

Since the cubic term is negative, we obtain

$$dr_t^i \leq \left[(1+\gamma)r_t^i + \left|\frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right| + \left|\frac{1}{N}\sum_{j=1}^N K_C(Z_t^i - Z_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\right|\right] dt.$$

Difference of the drifts : Let us now consider these last two terms

$$\left|\frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right| \leq \left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right| + \left|\frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j) - \frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j)\right|.$$

The first sum can be decomposed, using Assumption 4.1.1,

$$\begin{aligned}
\frac{1}{N}\left|\sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X(\bar{Z}_t^i - \bar{Z}_t^j)\right| &\leq \frac{L_X}{N}\sum_{j=1}^N \|Z_t^i - Z_t^j - (\bar{Z}_t^i - \bar{Z}_t^j)\|_1 \\
&\leq \frac{L_X}{N}\sum_{j=1}^N \left(\|Z_t^i - \bar{Z}_t^i\|_1 + \|Z_t^j - \bar{Z}_t^j\|_1\right) \\
&\leq L_X r_t^i + \frac{L_X}{N}\sum_{j=1}^N r_t^j.
\end{aligned}$$

Similarly, we obtain

$$\left|\frac{1}{N}\sum_{j=1}^N K_C(Z_t^i - Z_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\right| \leq L_C r_t^i + \frac{L_C}{N}\sum_{j=1}^N r_t^j + \left|\frac{1}{N}\sum_{j=1}^N K_C(\bar{Z}_t^i - \bar{Z}_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\right|.$$

Hence, we get

$$\begin{aligned}
dr_t^i \leq \Bigg[(1+\gamma)r_t^i + (L_X + L_C)\left(r_t^i + \frac{1}{N}\sum_{j=1}^N r_t^j\right) \\
+ \left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right| \\
+ \left|\frac{1}{N}\sum_{j=1}^N K_C(\bar{Z}_t^i - \bar{Z}_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\right|\Bigg] dt.
\end{aligned}$$

By considering the expectation, since $\mathbb{E}(r_t^j) = \mathbb{E}(r_t^i)$ for each j, by exchangeability of the particles, we have

$$\begin{aligned} d\mathbb{E}(r_t^i) \leq \Big(&(1+\gamma+2L_X+2L_C)\mathbb{E}(r_t^i) \\ &+ \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|\right] \\ &+ \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N K_C(\bar{Z}_t^i - \bar{Z}_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\right|\right]\Big)\, dt. \end{aligned}$$

Now, we bound the interaction part. We begin with K_X. By Cauchy-Schwarz, we can write

$$\begin{aligned} &\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|\right] \\ &\leq \mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|^2\right)^{1/2} \end{aligned}$$

We notice that $(\bar{Z}_t^j)_j$ are i.i.d with law $\bar{\mu}_t$. Let's denote $\bar{Z}_t$ a generic random variable of law $\bar{\mu}_t$ independent of $\bar{Z}_t^i$. What is more,

$$K_X * \bar{\mu}_t(\bar{Z}_t^i) = \int K_X(\bar{Z}_t^i - z)\bar{\mu}_t(dz) = \mathbb{E}[K_X(\bar{Z}_t^i - \bar{Z}_t)|\bar{Z}_t^i].$$

Hence

$$\begin{aligned} &\mathbb{E}\left(\mathbb{E}\left(\left|\frac{1}{N-1}\sum_{j\neq i} K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|^2 \Big|\bar{Z}_t^i\right)\right) \\ &= \mathbb{E}\left(\mathrm{Var}\left(\frac{1}{N-1}\sum_{j\neq i} K_X(\bar{Z}_t^i - \bar{Z}_t^j)\Big|\bar{Z}_t^i\right)\right) \\ &= \frac{1}{N-1}\mathbb{E}\left(\mathrm{Var}\left(K_X(\bar{Z}_t^i - \bar{Z}_t)\Big|\bar{Z}_t^i\right)\right) \\ &\leq \frac{L_X^2}{N-1}\mathbb{E}\left(\mathrm{Var}\left(\|\bar{Z}_t^i - \bar{Z}_t\|_1\Big|\bar{Z}_t^i\right)\right). \end{aligned}$$

Since

$$\begin{aligned} \mathbb{E}\left[\mathrm{Var}\left(\|\bar{Z}_t^i - \bar{Z}_t\|_1\Big|\bar{Z}_t^i\right)\right] &\leq \mathbb{E}\left[\mathbb{E}\left(\|\bar{Z}_t^i - \bar{Z}_t\|_1^2\Big|\bar{Z}_t^i\right)\right] \\ &\leq \mathbb{E}\left[\mathbb{E}\left(2\|\bar{Z}_t^i\|_1^2 + 2\|\bar{Z}_t\|_1^2\Big|\bar{Z}_t^i\right)\right] \leq 4\mathbb{E}(\|\bar{Z}_t\|_1^2), \end{aligned}$$

we obtain

$$\mathbb{E}\left(\mathbb{E}\left(\left|\frac{1}{N-1}\sum_{j\neq i} K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|^2 \Big|\bar{Z}_t^i\right)\right) \leq \frac{4L_X^2}{N-1}\mathbb{E}(\|\bar{Z}_t\|_1^2).$$

We now want to control $\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|^2\right)$. We decompose it with

$$
\begin{aligned}
&\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|^2\right)\\
=&\mathbb{E}\left(\left|\frac{N-1}{N}\frac{1}{N-1}\sum_{j=1}^N K_X(\bar{Z}_t^i-\bar{Z}_t^j)-\left(\frac{N-1}{N}+\frac{1}{N}\right)K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|^2\right)\\
\leq&2\left(\frac{N-1}{N}\right)^2\mathbb{E}\left(\left|\frac{1}{N-1}\sum_{j=1}^N K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|^2\right)\\
&+\frac{2}{N^2}\mathbb{E}\left(|K_X*\bar{\mu}_t(\bar{Z}_t^i)|^2\right).
\end{aligned}
$$

Since

$$
\begin{aligned}
\mathbb{E}\left(|K_X*\bar{\mu}_t(\bar{Z}_t^i)|^2\right)=&\mathbb{E}\left(\left|\mathbb{E}\left(K_X(\bar{Z}_t^i-\bar{Z}_t)|\bar{Z}_t^i\right)\right|^2\right)\\
&\leq L_X^2\mathbb{E}\left(\mathbb{E}\left(\|\bar{Z}_t^i-\bar{Z}_t\|_1^2|\bar{Z}_t^i\right)\right)\leq 4L_X^2\mathbb{E}(\|\bar{Z}_t\|_1^2),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\mathbb{E}&\left(\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|^2\right)\\
&\leq\left(\frac{N-1}{N}\right)^2\frac{4L_X^2}{N-1}\mathbb{E}(\|\bar{Z}_t\|_1^2)+\frac{4L_X^2}{N^2}\mathbb{E}(\|\bar{Z}_t\|_1^2)\\
&\leq\frac{8L_X^2}{N}\mathbb{E}(\|\bar{Z}_t\|_1^2),
\end{aligned}
\tag{4.1.9}
$$

and finally

$$
\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|\right]\leq\left(\frac{8L_X^2}{N}\mathbb{E}(\|\bar{Z}_t\|_1^2)\right)^{1/2}.
$$

Similarly, we have

$$
\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N K_C(\bar{Z}_t^i-\bar{Z}_t^j)-K_C*\bar{\mu}_t(\bar{Z}_t^i)\right|\right]\leq\left(\frac{8L_C^2}{N}\mathbb{E}(\|\bar{Z}_t\|_1^2)\right)^{1/2}.
$$

Finally,

$$
d\mathbb{E}(r_t^i)\leq\left((1+\gamma+2L_X+2L_C)\mathbb{E}(r_t^i)+\sqrt{8L_X^2+8L_C^2}\left(\frac{1}{N}\mathbb{E}(\|\bar{Z}_t\|_1^2)\right)^{1/2}\right)dt.
$$

Then by Proposition 4.1.6, there is a constant C_0 depending on initial conditions such that

$$d\mathbb{E}(r_t^i) \leq \left((1+\gamma+2L_X+2L_C)\mathbb{E}(r_t^i) + \frac{\sqrt{8L_X^2+8L_C^2}\sqrt{C_{0,1}}}{\sqrt{N}} e^{\frac{1}{2}C_{0,2}t} \right) dt$$

Conclusion : We have thus obtained

$$\begin{aligned} &d\left(\mathbb{E}(r_t^i) + \sqrt{\frac{8(L_X^2+L_C^2)C_{0,1}}{N}} \frac{1}{1+\gamma+2L_X+2L_C-\frac{C_{0,2}}{2}} e^{\frac{1}{2}C_{0,2}t} \right) \\ &\leq (1+\gamma+2L_X+2L_C) \\ &\quad \times \left(\mathbb{E}(r_t^i) \sqrt{\frac{8(L_X^2+L_C^2)C_{0,1}}{N}} \frac{1}{1+\gamma+2L_X+2L_C-\frac{C_{0,2}}{2}} e^{\frac{1}{2}C_{0,2}t} \right) dt \end{aligned}$$

and Gronwall's lemma yields

$$\begin{aligned} &\mathbb{E}(r_t^i) + \sqrt{\frac{8(L_X^2+L_C^2)C_{0,1}}{N}} \frac{1}{1+\gamma+2L_X+2L_C-\frac{C_{0,2}}{2}} e^{\frac{1}{2}C_{0,2}t} \\ &\leq e^{(1+\gamma+2L_X+2L_C)t} \left[\mathbb{E}(r_0^i) + \sqrt{\frac{8(L_X^2+L_C^2)C_{0,1}}{N}} \frac{1}{1+\gamma+2L_X+2L_C-\frac{C_{0,2}}{2}} \right], \end{aligned}$$

thus

$$\mathbb{E}(r_t^i) \leq C_1 e^{C_2 t} \frac{1}{\sqrt{N}}.$$

Let μ_0 a measure on $\mathbb{R}^2$, $\mu_t^{k,N}$ the marginal distribution at time t of the first k neurons $\left(Z_t^1, \ldots, Z_t^k\right)$ of an N particles system (4.1.1) with initial distribution $(\mu_0)^{\otimes N}$, and $\bar{\mu}_t$ is a solution of (4.1.2) with initial distribution μ_0.
We obtain for Wasserstein 1 distance

$$\begin{aligned} \mathcal{W}_1(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}) =& \inf \left\{ \mathbb{E}[\|Z^{(k)} - \bar{Z}^{(k)}\|_1], \mathbb{P}_{Z^{(k)}} = \mu_t^{k,N}, \mathbb{P}_{\bar{Z}^{(k)}} = \bar{\mu}_t^{\otimes k} \right\} \\ \leq& \inf \left\{ \mathbb{E}\left[\sum_{i=1}^k r_t^i \right], \mathbb{P}_{(Z_t^{i,N})_i} = \mu_t^{k,N}, \mathbb{P}_{(\bar{Z}_t^i)_i} = \bar{\mu}_t^{\otimes k} \right\} \\ \leq& k\mathbb{E}(r_t^1) \\ \leq& C_1 e^{C_2 t} \frac{k}{\sqrt{N}}. \end{aligned}$$

Hence Theorem 4.1.3.

4.2 Preliminaries

Index

Throughout this article, we define many parameters and constants. For the sake of clarity, we list the main ones here so as to give the reader an index to refer to.

— X, C, Z : X and C are the processes we consider (see (4.1.1) and (4.1.2)) and we often refer to $Z = (X, C)$,
— $\bar{\mu}_t = \mathrm{Law}(\bar{Z}_t)$: the density of the non linear limit (see (4.1.2)),
— α, β, γ, σ_X, σ_C : parameters of the problem (see (4.1.1)),
— $K_X, K_C, L_X, L_C, L_{X,max}, L_{C,max}$: K_X (resp. K_C) is an Lipschitz continuous interaction kernel, with Lipschitz constant $L_X \in [0, L_{X,max}]$ (resp. $L_C \in [0, L_{C,max}]$), as given in Assumption 4.1.1. In the case of uniform in time propagation of chaos, the inequalities L_X and L_C must satisfy are listed in Assumption 4.2.8,
— $\mathcal{W}_p$: the usual Wasserstein distance associated to the L^p distance (see (4.1.3)),
— a, $\tilde{a}$, $\mathcal{C}^0$: constants used to give an exponential initial moment to the problem (see the assumptions of Theorem 4.1.4 and Section 4.2.3),
— $\lambda, B, \tilde{B}, H, \tilde{H}, \alpha_X, \alpha_C, \beta_X, \beta_C$: H (resp. $\tilde{H}$) is a Lyapunov functions given in (4.2.2) (resp. (4.2.9)). Its main property involves parameters λ and B (resp. λ and $\tilde{B}$), as can for instance be seen in (4.2.4) (resp. (4.2.14)). $\alpha_X, \alpha_C, \beta_X$ and β_C are intermediate constants given in Lemma 4.2.3,
— c : a contraction rate (see Assumption 4.2.8),
— $r, f, g, \phi, \Phi, G, \rho, \delta, R, \epsilon, \mathcal{C}_1^{(f)}, \mathcal{C}_2^{(f)}$: f (see (4.2.22)) is a concave function, the definition of which involves g, ϕ, Φ (see Assumption 4.2.8 for these last three). Function G (see (4.2.27)) is then used to define ρ (see (4.2.26)), the semimetrics we consider in the end. All those notations thus refer to the modified distance we consider. These functions will be applied to r a modification of the usual L^1 distance (see equation (4.2.25)). Then, parameters δ, R, ϵ, $\mathcal{C}_1^{(f)}$, and $\mathcal{C}_2^{(f)}$ are used to define such functions (see Assumption 4.2.8 for some explicit values),
— R_0, ϕ_{min} : intermediate constants (see Assumption 4.2.8),
— $\mathcal{C}_0$: uniform in time bound on the second moment of the processes (see Lemma 4.2.5),
— $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_z$: constants used to quantify the control our modify distance has over the usual L^1 and L^2 distance (see Lemma 4.2.10 for the control and Assumption 4.2.8 for explicit values),
— $\phi_{rc}, \phi_{sc}, \xi$: ϕ_{rc} and ϕ_{sc} are two Lipschitz continuous functions used to define the coupling method, and their definitions involve a parameter ξ which converges to 0 in the end (see the beginning of Section 4.3).

4.2.1 Notations

For $h : \mathbb{R}^{2N} \to \mathbb{R}$, for all $(z_i)_{1\leq i\leq N} = (x_i, c_i)_{1\leq i\leq N} \in \mathbb{R}^{2N}$, the generator of (4.1.1) is $\mathcal{L}^N : \mathcal{C}^2(\mathbb{R}^{2N}, \mathbb{R}) \to \mathcal{C}^0(\mathbb{R}^{2N}, \mathbb{R})$, given by

$$\mathcal{L}^N h(z_1, .., z_N) = \sum_{i=1}^{N} \mathcal{L}^{i,N} h,$$

where for each $i \leq N$, $\mathcal{L}^{i,N} : \mathcal{C}^2(\mathbb{R}^{2N}, \mathbb{R}) \to \mathcal{C}^0(\mathbb{R}^{2N}, \mathbb{R})$ is defined as

$$\begin{aligned}\mathcal{L}^{i,N}h(z_1,..,z_N) = &\left(x_i - x_i^3 - c_i - \alpha + \frac{1}{N}\sum_{j=1}^N K_X(z_i - z_j)\right)\partial_{x_i}h\\ &+ \left(\gamma x_i - c_i - \beta + \frac{1}{N}\sum_{j=1}^N K_C(z_i - z_j)\right)\partial_{c_i}h\\ &+ \frac{\sigma_x^2}{2}\partial^2_{x_i,x_i}h + \frac{\sigma_c^2}{2}\partial^2_{c_i,c_i}h.\end{aligned}$$

For $h : \mathbb{R}^2 \to \mathbb{R}$, for all $z \in \mathbb{R}^2$, the generator of (4.1.2) for a given distribution μ is denoted by $\mathcal{L}_\mu : \mathcal{C}^2(\mathbb{R}^{2N}, \mathbb{R}) \to \mathcal{C}^0(\mathbb{R}^{2N}, \mathbb{R})$:

$$\begin{aligned}\mathcal{L}_\mu h(x,c) = &\big(x - x^3 - c - \alpha + K_X * \mu(z)\big)\,\partial_x h + (\gamma x - c - \beta + K_C * \mu(z))\,\partial_c h\\ &+ \frac{\sigma_x^2}{2}\partial^2_{xx}h + \frac{\sigma_c^2}{2}\partial^2_{cc}h.\end{aligned}$$

Remark 4.2.1. In particular, we notice that for fixed $(z_i)_{1\leq i\leq N} \in (\mathbb{R}^2)^N$, if we consider the empirical measure $\mu_{\text{emp}} = \frac{1}{N}\sum_j \delta_{z_j}$, we have for all $h : \mathbb{R}^2 \to \mathbb{R}$ and $\bar{z} \in \mathbb{R}^2$,

$$\begin{aligned}\mathcal{L}_{\mu_{\text{emp}}}h(\bar{z}) = &\left(\bar{x} - \bar{x}^3 - \bar{c} - \alpha + \frac{1}{N}\sum_{j=1}^N K_X(\bar{z} - z_j)\right)\partial_x h\\ &+ \left(\gamma\bar{x} - \bar{c} - \beta + \frac{1}{N}\sum_{j=1}^N K_C(\bar{z} - z_j)\right)\partial_c h + \frac{\sigma_x^2}{2}\partial^2_{xx}h + \frac{\sigma_c^2}{2}\partial^2_{cc}h.\end{aligned}$$

In this case, if we consider $\bar{z} = z_i$ for a specific i and we denote $\bar{h}^i : (z_1,..,z_N) \to h(z_i)$, then

$$\mathcal{L}_{\mu_{\text{emp}}}h(z_i) = \mathcal{L}^{i,N}\bar{h}^i(z_1,\ldots,z_N). \tag{4.2.1}$$

4.2.2 First Lyapunov function

We define a first Lyapunov function H. It won't be sufficient to control the dynamic in order to have uniform in time propagation of chaos, but it gives us a control on second moment of $(Z_t^{i,N})_i$ and $\bar{Z}_t$ with Propositions 4.1.5 and 4.1.6. Then, we will define in Subsection 4.2.3 a modified Lyapunov function.

Let $H : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$H(z) = H(x,c) = \frac{1}{2}\gamma x^2 + \beta x + \frac{1}{2}c^2 + \alpha c + H_0, \tag{4.2.2}$$

with

$$H_0 = \frac{\beta^2}{\gamma} + \alpha^2,$$

where γ, β and α are the parameters of the system (4.1.1). $H(z)$ can also be written

$$H(z) = \frac{\gamma}{2}\left(x + \frac{\beta}{\gamma}\right)^2 + \frac{1}{2}(c+\alpha)^2 + \frac{H_0}{2}.$$

Lemma 4.2.2. We have

(i) For all $x, c \in \mathbb{R}$, we have $H(x,c) \geq \frac{\gamma}{4}x^2 + \frac{c^2}{4} \geq 0$ and $H(x,c) \leq \gamma x^2 + c^2 + \frac{3}{2}H_0$,

(ii) For all $x, c \in \mathbb{R}$, we have $H(x,c) \geq \frac{1}{2\max(\gamma,1)}\left((\gamma x+\beta)^2 + (c+\alpha)^2\right)$,

(iii) For all $\delta > 0$ there is $\mathcal{C}_{r,H} > 0$ such that for all $x, x', c, c' \in \mathbb{R}$, we have

$$\left(|x-x'| + \delta|c-c'|\right)^2 \leq \mathcal{C}_{r,H}(H(x,c) + H(x',c')),$$

(iv) A direct consequence of the previous point is that for all $B \in \mathbb{R}$, $\lambda > 0$ and $\delta > 0$, there is $R \geq 0$ such that, for $x, x', c, c' \in \mathbb{R}$ satisfying $|x-x'| + \delta|c-c'| \geq R$, we have $H(z) + H(z') \geq \frac{64B}{15\lambda}$. An explicit value of R is given by $R = \sqrt{\frac{1024(1+\delta^2)B}{15\lambda\min(\gamma,1)}}$.

The first two points are consequences of direct calculations. The last two points are proved in Appendix 4.A.1.

Lemma 4.2.3 (Lyapunov's property of H). H, defined in (4.2.2), is a Lyapunov function. In fact, let $\lambda \in \mathbb{R}$ such that

$$\frac{L_X}{8} + L_C\left(2 + \frac{3}{32}\right) < 1 - \frac{\lambda}{2}, \tag{4.2.3}$$

then there exists $B > 0$ such that for all $(\bar{x}, \bar{c}) \in \mathbb{R}^2$, for all probability distribution μ on $\mathbb{R}^2$,

$$\begin{aligned}\mathcal{L}_\mu H(\bar{z}) \leq & B + (\alpha_X L_X + \beta_X L_C)\left(\mathbb{E}_\mu(|X|)^2 - \bar{x}^2\right) \\ & + (\alpha_C L_X + \beta_C L_C)\left(\mathbb{E}_\mu(|C|)^2 - \bar{c}^2\right) - \lambda H(\bar{z}).\end{aligned} \tag{4.2.4}$$

Moreover, for all $(z_i)_{1\leq i\leq N} \in \mathbb{R}^{2N}$, by denoting $H : (z_1, \ldots, z_N) \mapsto H(z_i)$,

$$\begin{aligned}\mathcal{L}^{i,N} H(z_1, \ldots, z_N) \leq & B + (\alpha_X L_X + \beta_X L_C)\left(\left(\frac{1}{N}\sum_{j=1}^N |x_j|\right)^2 - x_i^2\right) \\ & + (\alpha_C L_X + \beta_C L_C)\left(\left(\frac{1}{N}\sum_{j=1}^N |c_j|\right)^2 - c_i^2\right) - \lambda H(z_i),\end{aligned} \tag{4.2.5}$$

with

$$\alpha_X = \frac{\gamma}{2} + \frac{1}{2}, \quad \beta_X = \frac{17}{2}, \quad \alpha_C = \frac{1}{16}, \quad \beta_C = \frac{1}{2} + \frac{1}{32}$$

We refer to Appendix 4.A.2 for the proof of this lemma and of the following Proposition.

Proposition 4.2.4. A consequence of (4.2.4) is

$$\frac{d}{dt}\mathbb{E}H\left(\bar{Z}_t^i\right) \leq B - \lambda \mathbb{E}H\left(\bar{Z}_t^i\right). \tag{4.2.6}$$

Likewise, there are two main consequences to (4.2.5) that we will use :

$$\frac{d}{dt}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}H\left(Z_t^{i,N}\right)\right) \leq B - \lambda\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}H\left(Z_t^{i,N}\right), \tag{4.2.7}$$

and

$$\mathcal{L}^N\left(\frac{1}{N}\sum_{i=1}^{N}H\left(Z_t^{i,N}\right)\right) \leq B - \lambda\left(\frac{1}{N}\sum_{i=1}^{N}H\left(Z_t^{i,N}\right)\right), \tag{4.2.8}$$

We obtain Proposition 4.1.5 on the second moments of $X_t^{i,N}$ and $C_t^{i,N}$ thanks to the Inequation (4.2.7). We obtain the corresponding Proposition 4.1.6 for $\bar{X}$ and $\bar{C}$ with Inequation (4.2.6). The proof of these propositions is given in Appendix 4.A.2. It also yields the following uniform in time bound on the second moments when there exists $\lambda > 0$ satisfying (4.2.3).

Lemma 4.2.5. Provided the interaction kernels satisfy

$$\frac{L_X}{8} + L_C\left(2 + \frac{3}{32}\right) < 1,$$

if $\mathbb{E}(|\bar{X}_0|^2) < +\infty$ and $\mathbb{E}(|\bar{C}_0|^2) < +\infty$, then there exists C_0 such that for all $t \geq 0$:

$$\mathbb{E}\left(|\bar{X}_t^i|^2 + |\bar{C}_t^i|^2\right) \leq C_0.$$

From now on, we consider $\lambda > 0$ satisfying (4.2.3) (and use the *a priori* bounds $L_{X,max}$ and $L_{C,max}$ to ensure the existence of such a λ).

4.2.3 Modification of the function

Let $\tilde{\mathcal{C}}^0 > 0$, $\tilde{a} > 0$ and consider an initial measure μ_0 on $\mathbb{R}^2$ which satisfies $\mathbb{E}_{\mu_0}\left(e^{\tilde{a}(|X|+|C|)}\right) \leq \tilde{\mathcal{C}}^0$.

For technical reasons, we need a greater restoring force by the Lyapunov function than the one given in Lemma 4.2.3. We will need better control of the noise and interaction, in order to have uniform in time propagation of chaos. We thus modify it in order to obtain estimates such as (4.2.14) and (4.2.20) below.

Let $a > 0$, such that $a \leq \tilde{a}/\left(2\sqrt{2}\max\left(\sqrt{\gamma}, 1\right)\right)$. This choice of a is only necessary for further Propositions and Lemmas, in Section 4.3.

Let us consider for all $z \in \mathbb{R}^2$,

$$\tilde{H}(z) = \int_0^{H(z)} \exp\left(a\sqrt{u}\right) du = \frac{2}{a^2}\exp\left(a\sqrt{H(z)}\right)\left(a\sqrt{H(z)} - 1\right) + \frac{2}{a^2}. \tag{4.2.9}$$

Direct calculations yield the following technical lemma.

Lemma 4.2.6. We have, for all $z \in \mathbb{R}^2$

$$H(z) \exp\left(a\sqrt{H(z)}\right) \geq \tilde{H}(z) \geq \exp\left(a\sqrt{H(z)}\right) - \frac{2}{a^2}\left(\exp\left(\frac{a^2}{2}\right) - 1\right), \tag{4.2.10}$$

$$\frac{2}{a}\sqrt{H(z)} \exp\left(a\sqrt{H(z)}\right) \geq \tilde{H}(z) \geq \frac{1}{a}\sqrt{H(z)} \exp\left(a\sqrt{H(z)}\right) - \frac{1}{a^2}(e-2), \tag{4.2.11}$$

$$\tilde{H}(z) \geq H(z) \tag{4.2.12}$$

We may calculate, using Lemma 4.2.2 and Equation (4.2.4)

$$\begin{aligned}
\mathcal{L}_\mu\left(\tilde{H}\right) &= \exp\left(a\sqrt{H}\right)\mathcal{L}_\mu H + \frac{1}{2}\frac{a}{2\sqrt{H}}\exp\left(a\sqrt{H}\right)\left(|\sigma_x \partial_x H|^2 + |\sigma_c \partial_c H|^2\right) \\
&= \exp\left(a\sqrt{H}\right)\mathcal{L}_\mu H + \frac{a}{4\sqrt{H}}\exp\left(a\sqrt{H}\right)\left(\sigma_x^2(\gamma x + \beta)^2 + \sigma_c^2(c+\alpha)^2\right) \\
&\leq \exp\left(a\sqrt{H}\right)\left[B + (\alpha_X L_X + \beta_X L_C)\,\mathbb{E}_\mu(|X|)^2\right. \\
&\quad \left. + (\alpha_C L_X + \beta_C L_C)\,\mathbb{E}_\mu(|C|)^2 - \lambda H\right] \\
&\quad + \frac{1}{2}\max\left(\sigma_x^2, \sigma_c^2\right)\max(\gamma, 1)\, a\sqrt{H}\exp\left(a\sqrt{H}\right) \\
&\leq \exp\left(a\sqrt{H}\right)\left[B + \frac{\left(\frac{1}{2}\max\left(\sigma_x^2, \sigma_c^2\right)\max(\gamma,1)\right)^2 a^2}{2\lambda}\right. \\
&\quad \left. + (\alpha_X L_X + \beta_X L_C)\,\mathbb{E}_\mu(|X|)^2 + (\alpha_C L_X + \beta_C L_C)\,\mathbb{E}_\mu(|C|)^2 - \frac{\lambda}{2}H\right],
\end{aligned} \tag{4.2.13}$$

where for this last inequality we used Young's inequality

$$\frac{1}{2}\max\left(\sigma_x^2, \sigma_c^2\right)\max(\gamma, 1)\, a\sqrt{H} \leq \frac{\lambda}{2}H + \frac{\left(\frac{1}{2}\max\left(\sigma_x^2, \sigma_c^2\right)\max(\gamma,1)\right)^2 a^2}{2\lambda}.$$

Notice that (4.2.13) ensures that this new Lyapunov function also tends to bring back particle which ventured at infinity, and at an even greater rate. This new rate $H\exp(\sqrt{H})$ however comes at a cost : the initial condition must have a finite exponential moment, and no longer just have a finite second moment. First, by (4.2.6) and (4.2.12),

$$\mathbb{E}(\bar{X}_t)^2 \leq \frac{4}{\gamma}\mathbb{E}H\left(\bar{Z}_t\right) \leq \frac{4}{\gamma}\left(\frac{B}{\lambda} + \mathbb{E}H\left(\bar{Z}_0\right)\right) \leq \frac{4}{\gamma}\left(\frac{B}{\lambda} + \mathbb{E}\tilde{H}\left(\bar{Z}_0\right)\right) \leq \frac{4}{\gamma}\left(\frac{B}{\lambda} + \mathcal{C}^0\right).$$

Likewise, we have $\mathbb{E}(\bar{C}_t)^2 \leq 4\left(\frac{B}{\lambda} + \mathcal{C}^0\right)$. Furthermore, the function $h \mapsto \exp\left(a\sqrt{h}\right)\left(B - \frac{\lambda}{4}h\right)$ is bounded from above for $h \geq 0$. We therefore obtain from (4.2.13) the existence of $\tilde{B}$ such that

$$\mathcal{L}_{\bar{\mu}_t}\left(\tilde{H}\left(\bar{Z}_t^i\right)\right) \leq \tilde{B} - \frac{\lambda}{4}\left(H\left(\bar{Z}_t^i\right)\exp\left(a\sqrt{H\left(\bar{Z}_t^i\right)}\right)\right) \tag{4.2.14}$$

$$\frac{d}{dt}\mathbb{E}\tilde{H}\left(\bar{Z}_t^i\right) \leq \tilde{B} - \frac{\lambda}{4}\mathbb{E}\left(H\left(\bar{Z}_t^i\right)\exp\left(a\sqrt{H\left(\bar{Z}_t^i\right)}\right)\right) \tag{4.2.15}$$

$$\text{and} \quad \frac{d}{dt}\mathbb{E}\tilde{H}\left(\bar{Z}_t^i\right) \leq \tilde{B} - \frac{\lambda}{4}\mathbb{E}\tilde{H}\left(\bar{Z}_t^i\right), \tag{4.2.16}$$

where for this last inequality, we used (4.2.10). While (4.2.14) and (4.2.15) will be useful in ensuring a sufficient restoring force, Equation (4.2.16) give us a uniform in time bound on $\mathbb{E}\tilde{H}\left(\bar{Z}_t^i\right)$, provided we have an initial bound.

Now, for the system of particles, we have, using (4.2.13), $\forall i,\ \forall x_i, v_i \in \mathbb{R}^d$,

$$\begin{aligned}\mathcal{L}^N\tilde{H}(z_i) \leq \exp\left(a\sqrt{H(z_i)}\right)\Bigg(\tilde{B} + (\alpha_X L_X + \beta_X L_C)\left(\frac{1}{N}\sum_{j=1}^N |x_j|\right)^2 \\ + (\alpha_C L_X + \beta_C L_C)\left(\frac{1}{N}\sum_{j=1}^N |c_j|\right)^2 - \frac{\lambda}{2}H(z_i)\Bigg).\end{aligned}$$

Summing over $i \in \{1, .., N\}$, we may calculate

$$\begin{aligned}&(\alpha_X L_X + \beta_X L_C)\sum_{j=1}^N\left(\frac{\sum_{j=1}^N |x_j|}{N}\right)^2\sum_{i=1}^N\frac{\exp\left(a\sqrt{H(z_i)}\right)}{N} - \frac{\lambda}{16}\sum_{i=1}^N\frac{H(z_i)\exp\left(a\sqrt{H(z_i)}\right)}{N} \\ &\leq \frac{\lambda}{16}\Big(\sum_{i,j=1}^N\frac{H(z_i)}{N}\frac{\exp\left(a\sqrt{H(z_j)}\right)}{N} - \sum_{i=1}^N\frac{H(z_i)\exp\left(a\sqrt{H(z_i)}\right)}{N}\Big) \\ &\leq 0.\end{aligned} \tag{4.2.17}$$

Here, we used Lemma 4.2.2, the fact that $\forall x, y \geq 0\ \ xe^{\sqrt{y}} + ye^{\sqrt{x}} - xe^{\sqrt{x}} - ye^{\sqrt{y}} = (e^{\sqrt{x}} - e^{\sqrt{y}})(y - x) \leq 0$ and assumed

$$(\alpha_X L_X + \beta_X L_C) \leq \frac{\gamma\lambda}{64}.$$

Likewise,

$$\begin{aligned}(\alpha_C L_X + \beta_C L_C)\sum_{j=1}^N\left(\frac{\sum_{j=1}^N |c_j|}{N}\right)^2\sum_{i=1}^N\frac{\exp\left(a\sqrt{H(z_i)}\right)}{N} \\ - \frac{\lambda}{16}\sum_{i=1}^N\frac{H(z_i)\exp\left(a\sqrt{H(z_i)}\right)}{N} \leq 0,\end{aligned} \tag{4.2.18}$$

provided

$$(\alpha_C L_X + \beta_C L_C) \leq \frac{\lambda}{64}.$$

There is therefore a constant, which for the sake of clarity we will also denote $\tilde{B}$ (as we may take the maximum of the previous constants), such that we get

$$\begin{aligned}\mathcal{L}^{i,N}\tilde{H}(Z_t^{i,N}) \leq & \tilde{B} + (\alpha_X L_X + \beta_X L_C)\left(\frac{\sum_{j=1}^N |X_t^{j,N}|}{N}\right)^2 \exp\left(a\sqrt{H\left(Z_t^{i,N}\right)}\right) \\ & + (\alpha_C L_X + \beta_C L_C)\left(\frac{\sum_{j=1}^N |C_t^{j,N}|}{N}\right)^2 \exp\left(a\sqrt{H\left(Z_t^{i,N}\right)}\right) \\ & - \frac{\lambda}{4} H\left(Z_t^{i,N}\right)\exp\left(a\sqrt{H\left(Z_t^{i,N}\right)}\right) \qquad (4.2.19)\end{aligned}$$

$$\mathcal{L}^N\left(\frac{1}{N}\sum_{i=1}^N \tilde{H}(Z_t^{i,N})\right) \leq \tilde{B} - \frac{\lambda}{4}\left(\frac{1}{N}\sum_{i=1}^N H(Z_t^{i,N})\exp\left(a\sqrt{H\left(Z_t^{i,N}\right)}\right)\right) \qquad (4.2.20)$$

and

$$\mathcal{L}^N\left(\frac{1}{N}\sum_{i=1}^N \tilde{H}(Z_t^{i,N})\right) \leq \tilde{B} - \frac{\lambda}{4}\left(\frac{1}{N}\sum_{i=1}^N \tilde{H}(Z_t^{i,N})\right) \qquad (4.2.21)$$

Once again, (4.2.19) and (4.2.20) will be useful in ensuring a sufficient restoring force, and (4.2.21) will yield a uniform in time bound on the expectation of $\tilde{H}(Z_t^{i,N})$, described in the proof of Lemma 4.3.2 in the Appendix 4.A.5.

4.2.4 Parameters

We start by fixing the values of some parameters. The somewhat intricate expressions in this section are dictated by the computations arising in the proofs later on. They are somewhat roughly chosen and far from optimal as we only wish to convey the fact that every constant is explicit.

Recall α_X, β_X, α_C and β_C given in Lemma 4.2.3. $a > 0$ is fixed from the last Subsection and the definition of $\tilde{H}$ and $\tilde{B}$ are obtained from the same Subsection. Let $\lambda > 0$ be a fixed parameter satisfying

$$\frac{L_{X,max}}{8} + L_{C,max}\left(2 + \frac{3}{32}\right) < 1 - \frac{\lambda}{2}.$$

Remark 4.2.7. In fact, to have the existence of such a λ, we have to assume that

$$\frac{L_{X,max}}{8} + L_{C,max}\left(2 + \frac{3}{32}\right) < 1.$$

For example, we can consider $L_{X,max} = 4$ and $L_{C,max} = \frac{1}{5}$.

We consider the following parameters

Assumption 4.2.8. Given any $\eta > 15$ and $\tilde{\delta} > 0$, consider the following set of parameters

$$\begin{aligned}
\delta =&(1+\tilde{\delta})\frac{1+L_{X,max}}{1-L_{C,max}},\ R_0 = \sqrt{\frac{1024\tilde{B}}{15\lambda\min(\gamma,1)}},\ R=\sqrt{1+\delta^2}R_0,\\
\mathcal{C}_1^{(f)} =&16\gamma\left[\frac{1}{a^2}\left(1+a\left(\beta+\frac{\alpha}{\delta}\right)\sqrt{2\max(\gamma,1)}\right)\left(\exp\left(\frac{a^2}{2}\right)-1\right)\right.\\
&\left.+\sqrt{2\max(\gamma,1)}\left(\sqrt{\gamma}+\frac{1}{\delta}\right)(e-2)\right],\\
\mathcal{C}_2^{(f)} =&4\gamma\left[1+\left(a\left(\beta+\frac{\alpha}{\delta}\right)+2a^2\left(\sqrt{\gamma}+\frac{1}{\delta}\right)\right)\sqrt{2\max(\gamma,1)}\right],
\end{aligned}$$

$$\begin{aligned}
c =& \min\left\{\frac{2\tilde{B}}{\eta},\frac{\lambda}{32}\frac{\eta-15}{\eta},\frac{1}{1+\eta}\min\left(\frac{\sigma_x}{2\sqrt{\pi}R},\frac{\tilde{\delta}(1-L_{C,max})}{2(1+\tilde{\delta})}\right)\times\right.\\
&\left.\exp\left(-\frac{1}{4\sigma_x^2}\left(1+\delta\gamma+L_{X,max}+\delta L_{C,max}+\left(\mathcal{C}_1^{(f)}+\mathcal{C}_2^{(f)}\right)\sigma_x^2\right)R^2\right)\right\},\\
\epsilon =&\frac{\eta c}{2\tilde{B}},
\end{aligned}$$

$$\begin{aligned}
\phi_{min} &= \exp\left(-\frac{1}{4\sigma_x^2}(1+\delta\gamma+L_{X,max}+\delta L_{C,max}+(\epsilon\mathcal{C}_1^{(f)}+\mathcal{C}_2^{(f)})\sigma_x^2)R^2\right),\\
\mathcal{C}_1 &=\frac{1}{\min(\delta,1)}\frac{2}{\phi_{min}}\min\left(\frac{16\epsilon(1+\delta^2)}{\min(\gamma,1)},1\right),\\
\mathcal{C}_2 &=\frac{1}{\min(\delta^2,1)}\frac{2}{\phi_{min}}\min\left(\frac{16\epsilon(1+\delta^2)}{\min(\gamma,1)},1\right),\\
\mathcal{C}_z &=\frac{2}{\phi_{min}}\min\left(1,\frac{4}{\epsilon}\max\left(\sqrt{\frac{1}{\gamma}},1\right)\right).
\end{aligned}$$

We define f as follows

$$\begin{aligned}
f(r) &= \int_0^{r\wedge R}\phi(s)g(s)ds, \qquad (4.2.22)\\
\phi(r) &=\exp\left(-\frac{1}{4\sigma_x^2}\left(1+\delta\gamma+L_X+\delta L_C+\left(\epsilon\mathcal{C}_1^{(f)}+\mathcal{C}_2^{(f)}\right)\sigma_x^2\right)r^2\right),\\
\Phi(s) &= \int_0^s\phi(u)du,\\
g(r) &=1-\frac{c+2\epsilon\tilde{B}}{\sigma_x^2}\int_0^r\Phi(s)\phi(s)^{-1}ds.
\end{aligned}$$

Assume furthermore that L_X and L_C, the Lipschitz constants, satisfy

$$L_X \le \min\left(\frac{\lambda}{128\mathcal{C}_z},\frac{\lambda a}{512\epsilon\mathcal{C}_z},\frac{c}{2\mathcal{C}_1}\right) \quad \text{and} \quad L_C \le \min\left(\frac{\lambda}{128\delta\mathcal{C}_z},\frac{\lambda a}{512\epsilon\delta\mathcal{C}_z},\frac{c}{2\delta\mathcal{C}_1}\right). \qquad (4.2.23)$$

Since by Assumption 4.1.1, $L_X < L_{X,max}$ and $L_C < L_{C,max}$, we know

$$\frac{L_X}{8} + L_C\left(2+\frac{3}{32}\right) < 1 - \frac{\lambda}{2}.$$

By (4.2.23) and by considering greater $\mathcal{C}_z$ for instance, and using $L_X \leq \frac{\lambda}{128\mathcal{C}_z}$ and $L_C \leq \frac{\lambda}{128\delta\mathcal{C}_z}$, we have

$$\alpha_X L_X + \beta_X L_C \leq \frac{\gamma\lambda}{128} \quad \text{and} \quad \alpha_C L_X + \beta_C L_C \leq \frac{\lambda}{128}.$$

Obviously, these constants can be optimized.

We can define $\mathcal{C}_{K_X,K_C}$ of the Theorem 4.1.4 by:

$$\mathcal{C}_{K_X,K_C} \leq \frac{1}{\max(\delta,1)} \min\left(\frac{\lambda}{128\mathcal{C}_z}, \frac{\lambda a}{512\epsilon\mathcal{C}_z}, \frac{c}{2\mathcal{C}_1}\right). \tag{4.2.24}$$

Thus, if (4.2.24) is satisfied, then (4.2.23) is satisfied.

Notice how the bounds on L_X and L_C depend on c. This is one of the reasons why we use the *a priori* bounds $L_X \in [0, L_{X,max}]$ and $L_C \in [0, L_{C,max}]$ given in Assumption 4.1.1 : they allow us to bound c and δ independently of L_C and L_X. We are thus able to begin by choosing an acceptable values for those parameters, before then giving upper bounds on L_X and L_C.

We quickly mention that the constants $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_z$ above come from Lemma 4.2.10 later. We gather some properties required in the calculations of the proof of Theorem 4.1.4 in the following lemma. Again, these properties are the ones motivating the choice of parameters

Lemma 4.2.9. The set of parameters given in Assumption 4.2.8 satisfy

— f is $\mathcal{C}^2$ on $(0,R)$ such that $f'_+(0) = 1$ and $f'_-(R) > 0$, and constant on $[R,\infty)$. Moreover, we assume f is non-negative, non-decreasing and concave, and for all $s \geq 0$,

$$\min(s,R)\, f'_-(R) \leq f(s) \leq \min(s, f(R)) \leq \min(s,R).$$

— For all $r \in [0,R]$, $\phi(r) \geq \phi_{min}$ and $g(r) \geq \frac{1}{2}$.

— We have the conditions

$$2f'(R) \geq \exp\left(-\frac{1}{4\sigma_x^2}\left(1+\delta\gamma+L_X+\delta L_C+\left(\epsilon\mathcal{C}_1^{(f)}+\mathcal{C}_2^{(f)}\right)\sigma_x^2\right)R^2\right),$$

$$2c+4\epsilon\tilde{B} \leq \left(1-L_C-\frac{1+L_X}{\delta}\right)\min_{r\in]0,R]}\frac{f'(r)r}{f(r)},$$

$$c \leq \frac{\lambda}{32}\frac{\frac{64\epsilon\tilde{B}}{15\lambda}}{1+\frac{64\epsilon\tilde{B}}{15\lambda}}, \qquad \frac{1+L_X}{1-L_C} < \delta \quad \text{and} \quad \epsilon \leq 1.$$

The proof of this lemma is done in Appendix 4.A.3.

4.2.5 Control of the usual distances

The usual metrics, the L^1 distance and L^2 distance, are neither concave nor bounded. In consequence, we work with a semimetric $f(r(z,z'))$ between two particles $z \in \mathbb{R}^2$ and $z' \in \mathbb{R}^2$, where f has already been defined and r is defined as followed: let $\delta > \frac{1+L_X}{1-L_C} > 1$,

$$r(z,z') = r(x,c,x',c') = |x-x'| + \delta|c-c'|. \tag{4.2.25}$$

Let's notice that this semimetric is bounded and concave.

Then, in order to control the behavior of the particles, we use the structure of our systems, and consider the Lyapunov function of our particles. Thanks to this function, we will control the part of second order in the system. To study the distance between two particles systems, we work with $\rho((z_j,z'_j)_{1\le j\le N})$ defined as followed:

$$\rho\left((z_j,z'_j)_{1\le j\le N}\right) = \frac{1}{N}\sum_{i=1}^{N} f\left(r\left(z_i,z'_i\right)\right) G^i\left((z_j,z'_j)_j\right), \tag{4.2.26}$$

where for each $i \in \{1,...,N\}$,

$$G^i\left((z_j,z'_j)_j\right) = 1 + \epsilon\tilde{H}\left(z_i\right) + \epsilon\tilde{H}\left(z'_i\right) + \frac{\epsilon}{N}\sum_{j=1}^{N}\tilde{H}\left(z_j\right) + \frac{\epsilon}{N}\sum_{j=1}^{N}\tilde{H}\left(z'_j\right). \tag{4.2.27}$$

An immediate corollary of the definition and properties of H is that ρ is a quantity on $\mathbb{R}^{4N}$ which controls the usual L^1 and L2 distances. In particular, (i) and (iii) will be used to control $\mathcal{W}_1$.

Lemma 4.2.10. The constants $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_z > 0$, given in Assumption 4.2.8, are such that for all $z = (x,c) \in \mathbb{R}^2$ and $z' = (x',c') \in \mathbb{R}^2$

(i) $\|z-z'\|_1 \le \mathcal{C}_1 f\left(r\left(z,z'\right)\right)\left(1+\epsilon\tilde{H}(z)+\epsilon\tilde{H}(z')\right)$

(ii) $\|z-z'\|_2^2 \le \mathcal{C}_2 f\left(r\left(z,z'\right)\right)\left(1+\epsilon\tilde{H}(z)+\epsilon\tilde{H}(z')\right)$

(iii) $\|z-z'\|_1 \le \mathcal{C}_z f(r(z,z'))\left(1+\epsilon\sqrt{H(z)}+\epsilon\sqrt{H(z')}\right)$

The proof of this lemma is postponed to Appendix 4.A.4.

4.3 Proof of Theorem 4.1.4 in the case $\sigma_X > 0$

Let $\xi > 0$ be a parameter destined to vanish, and let $\varphi_{\text{sc}} : \mathbb{R}^+ \mapsto \mathbb{R}^+$ and $\varphi_{\text{rc}} : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be two Lipschitz continuous functions such that

$$\begin{aligned}
\forall x, \quad \varphi_{\text{sc}}^2(x) + \varphi_{\text{rc}}^2(x) =& 1 \\
\varphi_{\text{rc}}(x) =& 1 \text{ if } \xi \le x \le R \\
\varphi_{\text{rc}}(x) =& 0 \text{ if } x \le \frac{\xi}{2} \text{ or } x \ge R+\xi.
\end{aligned}$$

Intuitively, φ_{rc} represents the region of space in which we consider a reflection coupling (rc), and φ_{sc} the one in which we consider a synchronous coupling (sc). In fact, we will define

$$
\begin{aligned}
dB_t^{i,X} =&\varphi_{\mathrm{sc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)dB_t^{i,sc,X}+\varphi_{\mathrm{rc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)dB_t^{i,rc,X}\\
d\bar{B}_t^{i,X} =&\varphi_{\mathrm{sc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)dB_t^{i,sc,X}-\varphi_{\mathrm{rc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)dB_t^{i,rc,X}.
\end{aligned}
$$

In particular, we get $d(B_t^{i,X}-\bar{B}_t^{i,X})=2\varphi_{\mathrm{rc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)dB_t^{i,rc,X}$ which will be null if $x\notin\left[\frac{\xi}{2},R+\xi\right]$. The coupling is then symmetric. On $[\xi,R]$, the coupling is antithetic, with $d(B_t^{i,X}-\bar{B}_t^{i,X})=2dB_t^{i,rc,X}$.
We thus simultaneously construct the following solutions

$$
\begin{cases}
dX_t^{i,N}= & (X_t^{i,N}-(X_t^{i,N})^3-C_t^{i,N}-\alpha)dt+\frac{1}{N}\sum_{j=1}^N K_X(Z_t^i-Z_t^j)dt\\
& +\sigma_x\varphi_{\mathrm{sc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)dB_t^{i,sc,X}+\sigma_x\varphi_{\mathrm{rc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)dB_t^{i,rc,X}\\
dC_t^{i,N}= & (\gamma X_t^{i,N}-C_t^{i,N}+\beta)dt+\frac{1}{N}\sum_{j=1}^N K_C(Z_t^i-Z_t^j)dt+\sigma_c dB_t^{i,C},
\end{cases}
$$

and

$$
\begin{cases}
d\bar{X}_t^i= & (\bar{X}_t^i-(\bar{X}_t^i)^3-\bar{C}_t^i-\alpha)dt+K_X*\bar{\mu}_t(\bar{Z}_t^i)dt\\
& +\sigma_x\varphi_{\mathrm{sc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)dB_t^{i,sc,X}-\sigma_x\varphi_{\mathrm{rc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)dB_t^{i,rc,X}\\
d\bar{C}_t^i= & (\gamma\bar{X}_t^i-\bar{C}_t^i+\beta)dt+K_C*\bar{\mu}_t(\bar{Z}_t^i)dt+\sigma_c dB_t^{i,C}.
\end{cases}
$$

Notice that we consider a symmetric coupling on the dynamics of C.
In fact, note that the reflection is coordinate by coordinate, and isn't a global coupling as in [57].

4.3.1 Main proof and results

Proposition 4.3.1. We denote $r_t^i=r(Z_t^{i,N},\bar{Z}_t^i)$ and $G_t^i=G^i((Z_t^{j,N})_j,(\bar{Z}_t^j)_j)$, where G^i is defined in (4.2.27). For all $c\in\mathbb{R}$, for each $i\in\{1,\dots,N\}$, we have

$$
d(e^{ct}f(r_t^i)G_t^i)\le e^{ct}K_t^i dt+dM_t^i, \tag{4.3.1}
$$

where M_t^i is a continuous local martingale and K_t^i can be written as

$$
K_t^i=\tilde{K}_t^i+I_t^{1,i}+I_t^{2,i}+I_t^{3,i}. \tag{4.3.2}
$$

We define $\tilde{K}_t^i$, $I_t^{1,i}$, $I_t^{2,i}$, $I_t^{3,i}$ and $I_t^{4,i}$ as followed:

$$
\begin{aligned}
\tilde{K}_t^i =&G_t^i\Big[2cf(r_t^i)+f''(r_t^i)\left(2\sigma_x^2\varphi_{\mathrm{rc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)^2\right)\\
&+f'(r_t^i)\Big((1+\gamma\delta+L_X+\delta L_C)|X_t^{i,N}-\bar{X}_t^i|-|(X_t^{i,N})^3-(\bar{X}_t^i)^3|\\
&+(1+L_X+\delta L_C-\delta)|C_t^{i,N}-\bar{C}_t^i|+\left(\epsilon\mathcal{C}_1^{(f)}+\mathcal{C}_2^{(f)}\right)\sigma_x^2\varphi_{\mathrm{rc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)^2 r_t^i\Big)\Big]\\
&+\epsilon f(r_t^i)\left(4\tilde{B}-\frac{\lambda}{16}\tilde{H}(\bar{Z}_t^i)-\frac{\lambda}{16}\tilde{H}(Z_t^{i,N})-\frac{\lambda}{16N}\sum_{j=1}^N\tilde{H}(\bar{Z}_t^j)-\frac{\lambda}{16N}\sum_{j=1}^N\tilde{H}(Z_t^{j,N})\right),
\end{aligned} \tag{4.3.3}
$$

$$\begin{aligned}I_t^{1,i} =& G_t^i f'(r_t^i)\left(\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|\right)\\ &+ \delta G_t^i f'(r_t^i)\left(\left|\frac{1}{N}\sum_{j=1}^N K_C(\bar{Z}_t^i - \bar{Z}_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\right|\right),\end{aligned} \tag{4.3.4}$$

$$\begin{aligned}I_t^{2,i} =& G_t^i f'(r_t^i)\left(\frac{L_X}{N}\left(\sum_{j=1}^N \|Z_t^{j,N} - \bar{Z}_t^j\|_1\right)\right) + \delta G_t^i f'(r_t^i)\left(\frac{L_C}{N}\left(\sum_{j=1}^N \|Z_t^{j,N} - \bar{Z}_t^j\|_1\right)\right)\\ &- cf(r_t^i)G_t^i - \epsilon f(r_t^i)\left[\frac{\lambda}{16}H(\bar{Z}_t^i)\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right.\\ &+ \frac{\lambda}{16}H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right) + \frac{\lambda}{16N}\sum_{j=1}^N H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)\\ &\left.+\frac{\lambda}{16N}\sum_{j=1}^N H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right],\end{aligned} \tag{4.3.5}$$

and

$$\begin{aligned}I_t^{3,i} =& \epsilon f(r_t^i)\left((\alpha_X L_X + \beta_X L_C)\left(\frac{\sum_{j=1}^N |X_t^{j,N}|}{N}\right)^2 \exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right.\\ &+ (\alpha_C L_X + \beta_C L_C)\left(\frac{\sum_{j=1}^N |C_t^{j,N}|}{N}\right)^2 \exp\left(a\sqrt{H(Z_t^{i,N})}\right)\\ &- \frac{\lambda}{16}H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\\ &\left.-\frac{\lambda}{16N}\sum_{j=1}^N H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right).\end{aligned} \tag{4.3.6}$$

We need a control on $\mathbb{E}(G_t^i)$, which is a consequence of Lyapunov's properties on $\tilde{H}$ and the initial assumption of the Theorem 4.1.4. A proof is given in Appendix 4.3.7.

Lemma 4.3.2. There exists $\mathcal{C}_{G,1}$ and $\mathcal{C}_{G,2}$, such that for each $i \leq N$, for all $t > 0$, we have

$$\mathbb{E}(G_t^i) \leq \mathcal{C}_{G,1},$$
$$\mathbb{E}[(G_t^i)^2] \leq \mathcal{C}_{G,2}.$$

Each of the terms given in Proposition 4.3.1 will be controlled by different ways. The following Lemmas summarizes it. The first term, $\tilde{K}_t^i$, contains the various behaviors we have previously identified : we deal with it either through a synchronous coupling (when the deterministic drift is contracting), or through a reflection coupling (notice the second derivative f" which will provide contraction provided f is sufficiently concave). Finally, notice the effect of Lyapunov function $\tilde{H}$ which yields a restoring force.

Lemma 4.3.3. For each $i \leq N$, for all $t > 0$,

$$\mathbb{E}\tilde{K}_t^i \leq \xi \left(2 + \delta\gamma + L_X + \delta L_C - L_C - \frac{1 + L_X}{\delta} \right) \mathbb{E}G_t^i. \tag{4.3.7}$$

The interaction term $\frac{1}{N} K_X(Z_t^{j,N} - Z_t^{i,N}) - K_X * \bar{\mu}_t(\bar{Z}_t^i)$ can be decomposed into $\{\frac{1}{N} K_X(\bar{Z}_t^j - \bar{Z}_t^i) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\}$ and $\frac{1}{N} \sum \left[K_X(Z_t^{j,N} - Z_t^{i,N}) - K_X(\bar{Z}_t^j - \bar{Z}_t^i) \right]$. The first part is in $I_t^{1,i}$ is dealt with using some form of law of large number in a similar way as what has been done in the proof of Theorem 4.1.3.

Lemma 4.3.4. For each $i \leq N$, for all $t > 0$,

$$\mathbb{E}(I_t^{1,i}) \leq \sqrt{\frac{3}{N-1}} \mathcal{C}_{G,2}^{1/2} \sqrt{2C_0} (L_X + L_C), \tag{4.3.8}$$

where $\mathcal{C}_{G,2}$ is defined in Lemma 4.3.2 and C_0 is defined in Lemma 4.2.5.

$I_t^{2,i}$ contains the leftovers of this decomposition and of some of the additional terms of the Lyapunov function. It will be compensated in expectation.

Lemma 4.3.5. For all $t > 0$,

$$\frac{1}{N} \sum_{i=1}^{N} I_t^{2,i} \leq 0. \tag{4.3.9}$$

Finally, $I_t^{3,i}$ deals with the non linearity appearing in the dynamics of the Lyapunov function, and will be non positive for values of L_X and L_C sufficiently small. It is also here we justify adding the last two terms in (4.2.27).

Lemma 4.3.6. For each $i \leq N$, for all $t > 0$,

$$I_t^{3,i} \leq 0. \tag{4.3.10}$$

Proof of Theorem 4.1.4. With these four Lemmas, we can calculate

$$\begin{aligned}
\frac{1}{N} \sum_{i=1}^{N} \mathbb{E}K_t^i =& \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\tilde{K}_t^i + \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}I_t^{1,i} + \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}I_t^{2,i} + \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}I_t^{3,i} \\
\leq& \frac{1}{N} \sum_{i=1}^{N} \xi \left(2 + \delta\gamma + L_X + \delta L_C - L_C - \frac{1 + L_X}{\delta} \right) \mathbb{E}G_t^i \\
&+ \frac{1}{N} \sum_{i=1}^{N} \sqrt{\frac{3}{N-1}} \mathcal{C}_{G,2}^{1/2} \sqrt{2C_0} (L_X + L_C) \\
\leq& \xi \left(2 + \delta\gamma + L_X + \delta L_C - L_C - \frac{1 + L_X}{\delta} \right) \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}G_t^i \\
&+ \sqrt{\frac{3}{N-1}} \mathcal{C}_{G,2}^{1/2} \sqrt{2C_0} (L_X + L_C)
\end{aligned}$$

Since by Lemma 4.3.2, we have

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}G_t^i \leq \mathcal{C}_{G,1},$$

we obtain

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}K_t^i \leq \xi A + (L_X + L_C)\frac{B}{\sqrt{N}}$$

where A and B are constants.

For all initial couplings such that $\mathbb{E}\rho\left((Z_0^j,\bar{Z}_0^j)_{1\leq j\leq N}\right) < \infty$, by taking the expectation of (4.3.1) along a sequence of increasing stopping times, we have thanks to Fatou's lemma

$$\begin{aligned}
e^{ct}\mathbb{E}\left(\rho\left((Z_t^j,\bar{Z}_t^j)_{1\leq j\leq N}\right)\right) \leq& \mathbb{E}\left(\rho\left((Z_0^j,\bar{Z}_0^j)_{1\leq j\leq N}\right)\right) + \xi A\int_0^t e^{cs}ds \\
&+ (L_X+L_C)\frac{B}{\sqrt{N}}\int_0^t e^{cs}ds \\
\leq& \mathbb{E}\left(\rho\left((Z_0^j,\bar{Z}_0^j)_{1\leq j\leq N}\right)\right) + \xi A\frac{e^{ct}-1}{c} \\
&+ (L_X+L_C)\frac{B}{\sqrt{N}}\frac{e^{ct}-1}{c}.
\end{aligned}$$

We obtain

$$\begin{aligned}
\mathbb{E}\left(\rho\left((Z_t^j,\bar{Z}_t^j)_{1\leq j\leq N}\right)\right) \leq& \mathbb{E}\left(\rho\left((Z_0^j,\bar{Z}_0^j)_{1\leq j\leq N}\right)\right)e^{-ct} + \frac{\xi A}{c}\left(1-e^{-ct}\right) \\
&+ \frac{(L_X+L_C)B}{c}\frac{1}{\sqrt{N}}\left(1-e^{-ct}\right).
\end{aligned}$$

By using the exchangeability of the particles, we have

$$\mathbb{E}\left(\rho\left((Z_t^j,\bar{Z}_t^j)_{1\leq j\leq N}\right)\right) = \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}f(r_t^i)G_t^i\right) = \mathbb{E}\left(\frac{1}{k}\sum_{i=1}^{k}f(r_t^i)G_t^i\right)$$

for all $k\in\mathbb{N}$. Then

$$\mathbb{E}\left(\sum_{i=1}^{k}f(r_t^i)G_t^i\right) = k\mathbb{E}\left(\rho\left((Z_t^j,\bar{Z}_t^j)_{1\leq j\leq N}\right)\right).$$

Let μ_0 a measure on $\mathbb{R}^2$, $\mu_t^{k,N}$ the marginal distribution at time t of the first k neurons $\left((X_t^1,C_t^1),....,(X_t^k,C_t^k)\right)$ of an N particles system (4.1.1) with initial distribution $(\mu_0)^{\otimes N}$, and $\bar{\mu}_t$ is a solution of (4.1.2) with initial distribution μ_0. This implies

$\mathbb{E}\left(\rho\left((Z_0^j,\bar{Z}_0^j)_{1\leq j\leq N}\right)\right)=0$. By Lemma 4.2.10, we obtain for Wasserstein 1 distance

$$\begin{aligned}
\mathcal{W}_1(\mu_t^{k,N},\bar{\mu}_t^{\otimes k}) =& \inf\left\{\mathbb{E}[\|Z^{(k)}-\bar{Z}^{(k)}\|_1],\mathbb{P}_{Z^{(k)}}=\mu_t^{k,N},\mathbb{P}_{\bar{Z}^{(k)}}=\bar{\mu}_t^{\otimes k}\right\}\\
=& \inf\left\{\mathbb{E}\left[\sum_{i=1}^k\|Z_t^{i,N}-\bar{Z}_t^i\|_1\right],\mathbb{P}_{(Z_t^{i,N})_i}=\mu_t^{k,N},\mathbb{P}_{(\bar{Z}_t^i)_i}=\bar{\mu}_t^{\otimes k}\right\}\\
\leq& \inf\left\{\mathcal{C}_1\mathbb{E}\left[\sum_{i=1}^k f(r_t^i)G_t^i\right],\mathbb{P}_{(Z_t^{i,N})_i}=\mu_t^{k,N},\mathbb{P}_{(\bar{Z}_t^i)_i}=\bar{\mu}_t^{\otimes k}\right\}\\
\leq& \inf\left\{k\mathcal{C}_1\mathbb{E}\left(\rho\left((Z_t^j,\bar{Z}_t^j)_{1\leq j\leq N}\right)\right),\mathbb{P}_{(Z_t^{i,N})_i}=\mu_t^{k,N},\mathbb{P}_{(\bar{Z}_t^i)_i}=\bar{\mu}_t^{\otimes k}\right\}\\
\leq& k\mathcal{C}_1\inf\left\{\mathbb{E}\left(\rho\left((Z_0^j,\bar{Z}_0^j)_{1\leq j\leq N}\right)\right)+\frac{(L_X+L_C)B}{c}\frac{1}{\sqrt{N}},\right.\\
&\qquad\left.\mathbb{P}_{(Z_t^{i,N})_i}=\mu_t^{k,N},\mathbb{P}_{(\bar{Z}_t^i)_i}=\bar{\mu}_t^{\otimes k}\right\}\\
\leq& \frac{\xi Ak\mathcal{C}_1}{c}\left(1-e^{-ct}\right)+\frac{(L_X+L_C)B\mathcal{C}_1}{c}\frac{k}{\sqrt{N}}\left(1-e^{-ct}\right)
\end{aligned}$$

By taking the limit as $\xi\to 0$ uniformly in time, we obtain the desired result. The same lemma and the same type of calculations yield the result for Wasserstein 2

$$\mathcal{W}_2(\mu_t^{k,N},\bar{\mu}_t^{\otimes k})^2\leq\frac{k}{\sqrt{N}}\mathcal{C}_2\frac{(L_X+L_C)B}{c}.$$

□

4.3.2 Proof of the decomposition

Proof of Proposition 4.3.1. First, we need to calculate $d(e^{ct}f(r_t^i)G_t^i)$, where we recall

$$r_t^i=|X_t^{i,N}-\bar{X}_t^i|+\delta|C_t^{i,N}-\bar{C}_t^i|$$

and

$$G_t^i=1+\epsilon\tilde{H}(\bar{Z}_t^i)+\epsilon\tilde{H}(Z_t^{i,N})+\frac{\epsilon}{N}\sum_{j=1}^N\tilde{H}(Z_t^{j,N})+\frac{\epsilon}{N}\sum_{j=1}^N\tilde{H}(\bar{Z}_t^j).$$

We have already calculated $d(X_t^{i,N}-\bar{X}_t^i)$ and $d|X_t^{i,N}-\bar{X}_t^i|$ in the case of symmetric coupling in Subsection 4.1.4 in (4.1.7). Here, we need to use Ito's formula and dominated convergence theorem (see Lemma 3.1 of [40]) to consider the Brownian term. We obtain

$$d|X_t^{i,N}-\bar{X}_t^i|=A_t^Xdt+2\mathrm{sign}(X_t^{i,N}-\bar{X}_t^i)\sigma_x\varphi_{\mathrm{rc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)dB_t^{i,rc,X},$$

with

$$\begin{aligned}
A_t^X\leq&|X_t^{i,N}-\bar{X}_t^i|-\left|(X_t^{i,N})^3-(\bar{X}_t^i)^3\right|+\left|C_t^{i,N}-\bar{C}_t^i\right|\\
&+\left|\frac{1}{N}\sum_{j=1}^N K_X(Z_t^i-Z_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|.
\end{aligned}$$

Likewise, as it has already been calculated in (4.1.8) in Subsection 4.1.4,

$$d|C_t^{i,N} - \bar{C}_t^i| = A_t^C dt, \tag{4.3.11}$$

with

$$A_t^C \leq \gamma \left| X_t^{i,N} - \bar{X}_t^i \right| - |C_t^{i,N} - \bar{C}_t^i| + \left| \frac{1}{N} \sum_{j=1}^N K_C(Z_t^i - Z_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i) \right|.$$

Now we have

$$dr_t^i = \left(A_t^X + \delta A_t^C\right) dt + 2\text{sign}(X_t^{i,N} - \bar{X}_t^i)\sigma_x \varphi_{\text{rc}} \left(|X_t^{i,N} - \bar{X}_t^i|\right) dB_t^{i,rc,X}$$

and we deduce with the Ito's formula

$$df(r_t^i) = f'(r_t^i) dr_t^i + \frac{1}{2} f''(r_t^i) \left(2\sigma_x \varphi_{\text{rc}} \left(|X_t^{i,N} - \bar{X}_t^i|\right)\right)^2 dt.$$

Finally, for $c > 0$,

$$d(e^{ct} f(r_t^i)) = ce^{ct} f(r_t^i) dt + e^{ct} df(r_t^i).$$

Then, by Ito's formula,

$$\begin{aligned}
\frac{1}{\epsilon} dG_t^i = &\left(\mathcal{L}_{\bar{\mu}_t} \tilde{H}(\bar{Z}_t^i) + \mathcal{L}^N \tilde{H}(Z_t^{i,N})\right) dt \\
&+ \sigma_x \varphi_{\text{rc}} \left(|X_t^{i,N} - \bar{X}_t^i|\right) \left(\partial_x \tilde{H}(Z_t^{i,N}) - \partial_x \tilde{H}(\bar{Z}_t^i)\right) dB_t^{i,rc,X} \\
&+ \sigma_x \varphi_{\text{sc}} \left(|X_t^{i,N} - \bar{X}_t^i|\right) \left(\partial_x \tilde{H}(Z_t^{i,N}) + \partial_x \tilde{H}(\bar{Z}_t^i)\right) dB_t^{i,sc,X} \\
&+ \sigma_c \left(\partial_c \tilde{H}(Z_t^{i,N}) + \partial_c \tilde{H}(\bar{Z}_t^i)\right) dB_t^{i,C} \\
&+ \frac{1}{N} \sum_{j=1}^N \left(\mathcal{L}_{\bar{\mu}_t} \tilde{H}(\bar{Z}_t^j) + \mathcal{L}^N \tilde{H}(Z_t^{j,N})\right) dt \\
&+ \frac{\sigma_x}{N} \sum_{j=1}^N \varphi_{\text{rc}} \left(|X_t^{j,N} - \bar{X}_t^j|\right) \left(\partial_x \tilde{H}(Z_t^{j,N}) - \partial_x \tilde{H}(\bar{Z}_t^j)\right) dB_t^{j,rc,X} \\
&+ \frac{\sigma_x}{N} \sum_{j=1}^N \varphi_{\text{sc}} \left(|X_t^{j,N} - \bar{X}_t^j|\right) \left(\partial_x \tilde{H}(Z_t^{j,N}) + \partial_x \tilde{H}(\bar{Z}_t^j)\right) dB_t^{j,sc,X} \\
&+ \frac{\sigma_c}{N} \sum_{j=1}^N \left(\partial_c \tilde{H}(Z_t^{j,N}) + \partial_c \tilde{H}(\bar{Z}_t^j)\right) dB_t^{j,C}.
\end{aligned}$$

We finally get

$$\begin{aligned}
d(e^{ct} f(r_t^i) G_t^i) = &G_t^i d(e^{ct} f(r_t^i)) + e^{ct} f(r_t^i) dG_t^i + 2\epsilon \left(1 + \frac{1}{N}\right) \sigma_x^2 \varphi_{\text{rc}} \left(|X_t^{i,N} - \bar{X}_t^i|\right)^2 \\
&\times \text{sign}(X_t^{i,N} - \bar{X}_t^i) \left(\partial_x \tilde{H}(Z_t^{i,N}) - \partial_x \tilde{H}(\bar{Z}_t^i)\right) e^{ct} f'(r_t^i) dt.
\end{aligned}$$

Now, we need to use the following Lemma, proven in Appendix 4.A.5, to have a more tractable expression:

Lemma 4.3.7. We have the majoration

$$
\begin{aligned}
2\epsilon\left(1+\frac{1}{N}\right)\sigma_x^2\varphi_{\mathrm{rc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)^2\operatorname{sign}(X_t^{i,N}-\bar{X}_t^i)\left(\partial_x\tilde{H}(Z_t^{i,N})-\partial_x\tilde{H}(\bar{Z}_t^i)\right)\\
\leq\left(\epsilon\mathcal{C}_1^{(f)}+\mathcal{C}_2^{(f)}\right)\sigma_x^2\varphi_{\mathrm{rc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)^2 r_t^i G_t^i.
\end{aligned}
$$

Eventually, by denoting the terms in $dB_t^{i,rc,X}$, $dB_t^{i,sc,X}$, $dB_t^{i,C}$, ... as the local martingale dM_t^i, we obtain

$$
\begin{aligned}
d(e^{ct}f(r_t^i)G_t^i)\leq & G_t^i c e^{ct}f(r_t^i)dt+e^{ct}G_t^i f'(r_t^i)\left(A_t^X+\delta A_t^C\right)dt\\
&+e^{ct}G_t^i\frac{1}{2}f''(r_t^i)\left(2\sigma_x\varphi_{\mathrm{rc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)\right)^2dt\\
&+e^{ct}f(r_t^i)\left(\mathcal{L}_{\bar{\mu}_t}\tilde{H}(\bar{Z}_t^i)+\mathcal{L}^N\tilde{H}(Z_t^{i,N})\right)dt\\
&+e^{ct}f(r_t^i)\frac{1}{N}\sum_{j=1}^N\left(\mathcal{L}_{\bar{\mu}_t}\tilde{H}(\bar{Z}_t^j)+\mathcal{L}^N\tilde{H}(Z_t^{j,N})\right)dt\\
&+\left(\epsilon\mathcal{C}_1^{(f)}+\mathcal{C}_2^{(f)}\right)\sigma_x^2\varphi_{\mathrm{rc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)^2 r_t^i G_t^i e^{ct}f'(r_t^i)dt\\
&+dM_t^i.
\end{aligned}
$$

We use (4.2.14) to bound $\mathcal{L}_{\bar{\mu}_t}\tilde{H}(\bar{Z}_t^i)$ and (4.2.19) to bound $\mathcal{L}^N\tilde{H}(Z_t^{i,N})$. The interaction terms in A_t^X and A_t^C are decomposed and we define $I_t^{1,i}$ as follows

$$
\begin{aligned}
I_t^{1,i}= & G_t^i f'(r_t^i)\left(\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|\right)\\
&+\delta G_t^i f'(r_t^i)\left(\left|\frac{1}{N}\sum_{j=1}^N K_C(\bar{Z}_t^i-\bar{Z}_t^j)-K_C*\bar{\mu}_t(\bar{Z}_t^i)\right|\right).
\end{aligned}
$$

The second part of the decomposition is grouped in $I_t^{2,i}$ with compensating terms, and in particular Lyapunov functions, which appears with the use of (4.2.14) and

(4.2.19), to control the sum:

$$
\begin{aligned}
I_t^{2,i} =& G_t^i f'(r_t^i)\left(\frac{L_X}{N}\left(\sum_{j=1}^N |X_t^{j,N} - \bar{X}_t^j| + |C_t^{j,N} - \bar{C}_t^j|\right)\right) \\
&+ \delta G_t^i f'(r_t^i)\left(\frac{L_C}{N}\left(\sum_{j=1}^N |X_t^{j,N} - \bar{X}_t^j| + |C_t^{j,N} - \bar{C}_t^j|\right)\right) \\
&- cf(r_t^i)G_t^i - \epsilon f(r_t^i)\left[\frac{\lambda}{16}H(\bar{Z}_t^i)\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right. \\
&+ \frac{\lambda}{16}H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right) + \frac{\lambda}{16N}\sum_{j=1}^N H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right) \\
&\left.+\frac{\lambda}{16N}\sum_{j=1}^N H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right].
\end{aligned}
$$

We gather the expectations terms, obtained with (4.2.19), in $I_t^{3,i}$, and we keep a fraction of Lyapunov function to control it:

$$
\begin{aligned}
I_t^{3,i} = \epsilon f(r_t^i)&\left((\alpha_X L_X + \beta_X L_C)\left(\frac{\sum_{j=1}^N |X_t^{j,N}|}{N}\right)^2 \exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right. \\
&+ (\alpha_C L_X + \beta_C L_C)\left(\frac{\sum_{j=1}^N |C_t^{j,N}|}{N}\right)^2 \exp\left(a\sqrt{H(Z_t^{i,N})}\right) \\
&\left.-\frac{\lambda}{16}H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right) - \frac{\lambda}{16N}\sum_{j=1}^N H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right).
\end{aligned}
$$

Finally, we define $\tilde{K}_t^i$ with the leftovers. It will, in particular, give the constraints on f which explain its choice.

$$
\begin{aligned}
\tilde{K}_t^i =& G_t^i\Big[2cf(r_t^i) + f''(r_t^i)\left(2\sigma_x^2\varphi_{\mathrm{rc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right)^2\right) \\
&+ f'(r_t^i)\Big((1+\gamma\delta + L_X + \delta L_C)|X_t^{i,N} - \bar{X}_t^i| - |(X_t^{i,N})^3 - (\bar{X}_t^i)^3| \\
&+ (1 + L_X + \delta L_C - \delta)|C_t^{i,N} - \bar{C}_t^i| + \left(\epsilon\mathcal{C}_1^{(f)} + \mathcal{C}_2^{(f)}\right)\sigma_x^2\varphi_{\mathrm{rc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right)^2 r_t^i\Big)\Big] \\
&+ \epsilon f(r_t^i)\left(4\tilde{B} - \frac{\lambda}{16}\tilde{H}(\bar{Z}_t^i) - \frac{\lambda}{16}\tilde{H}(Z_t^{i,N}) - \frac{\lambda}{16N}\sum_{j=1}^N \tilde{H}(\bar{Z}_t^j) - \frac{\lambda}{16N}\sum_{j=1}^N \tilde{H}(Z_t^{j,N})\right),
\end{aligned}
$$

□

4.3.3 Controls of $I_t^{1,i}$, $I_t^{2,i}$ and $I_t^{3,i}$

Proof of Lemma 4.3.6. Since we assume

$$\frac{4}{\gamma}\left(\alpha_X L_X + \beta_X L_C\right) \leq \frac{\lambda}{32} \quad \text{and} \quad 4\left(\alpha_C L_X + \beta_C L_C\right) \leq \frac{\lambda}{32},$$

and since

$$H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right) \leq H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right) + H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)$$

we obtain

$$\begin{aligned}
&(\alpha_X L_X + \beta_X L_C)\left(\frac{\sum_{j=1}^N |X_t^{j,N}|}{N}\right)^2 \exp\left(a\sqrt{H(Z_t^{i,N})}\right) \\
&+ (\alpha_C L_X + \beta_C L_C)\left(\frac{\sum_{j=1}^N |C_t^{j,N}|}{N}\right)^2 \exp\left(a\sqrt{H(Z_t^{i,N})}\right) \\
&- \frac{\lambda}{16N}\left(NH(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right) + \sum_{j=1}^N H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right) \leq 0.
\end{aligned}$$

Then, for each $i \leq N$, and for all $t > 0$, $I_t^{3,i} \leq 0$. $\square$

Proof of Lemma 4.3.5. We prove the non-positivity of $\frac{1}{N}\sum_{i=1}^N I_t^{2,i}$, where we recall for each $1 \leq i \leq N$, for all $t > 0$

$$\begin{aligned}
I_t^{2,i} =& G_t^i f'(r_t^i)\left(\frac{L_X}{N}\left(\sum_{j=1}^N |X_t^{j,N} - \bar{X}_t^j| + |C_t^{j,N} - \bar{C}_t^j|\right)\right) \\
&+ \delta G_t^i f'(r_t^i)\left(\frac{L_C}{N}\left(\sum_{j=1}^N |X_t^{j,N} - \bar{X}_t^j| + |C_t^{j,N} - \bar{C}_t^j|\right)\right) \\
&- cf(r_t^i)G_t^i - \epsilon f(r_t^i)\left[\frac{\lambda}{16}H(\bar{Z}_t^i)\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right. \\
&+ \frac{\lambda}{16}H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right) + \frac{\lambda}{16N}\sum_{j=1}^N H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right) \\
&\left.+ \frac{\lambda}{16N}\sum_{j=1}^N H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right].
\end{aligned}$$

First, note

$$\begin{aligned}
\sum_{i=1}^N G_t^i =& N + \epsilon\sum_{i=1}^N\left(\tilde{H}\left(\bar{Z}_t^i\right) + \tilde{H}(Z_t^{i,N})\right) + \frac{\epsilon}{N}\sum_{i=1}^N\sum_{k=1}^N\left(\tilde{H}\left(\bar{Z}_t^k\right) + \tilde{H}(Z_t^{k,N})\right) \\
=& N + 2\epsilon\sum_{i=1}^N\left(\tilde{H}\left(\bar{Z}_t^i\right) + \tilde{H}(Z_t^{i,N})\right).
\end{aligned}$$

Since $f'\left(r_t^i\right) \leq 1$, we have

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{N}f'\left(r_t^i\right)G_t^i\sum_{j=1}^{N}\|Z_t^{j,N}-\bar{Z}_t^j\|_1\right)\\
\leq&\frac{1}{N^2}\left(\sum_{j=1}^{N}\|Z_t^{j,N}-\bar{Z}_t^j\|_1\right)\left(\sum_{i=1}^{N}G_t^i\right)\\
\leq&\frac{1}{N}\sum_{i=1}^{N}\|Z_t^{i,N}-\bar{Z}_t^i\|_1+\frac{2\epsilon}{N^2}\sum_{i,j=1}^{N}\|Z_t^{i,N}-\bar{Z}_t^i\|_1\left(\tilde{H}\left(\bar{Z}_t^j\right)+\tilde{H}(Z_t^{j,N})\right)
\end{aligned}
$$

and, using Lemma 4.2.10 (i)

$$
\frac{1}{N}\sum_{i=1}^{N}\|Z_t^{i,N}-\bar{Z}_t^i\|_1 \leq \frac{\mathcal{C}_1}{N}\sum_{i=1}^{N}f(r_t^i)G_t^i
$$

and with Lemma 4.2.10 (iii)

$$
\begin{aligned}
&\sum_{i,j=1}^{N}\|Z_t^{i,N}-\bar{Z}_t^i\|_1\left(\tilde{H}\left(\bar{Z}_t^j\right)+\tilde{H}(Z_t^{j,N})\right)\\
\leq&\mathcal{C}_z\sum_{i,j=1}^{N}f(r_t^i)\left(1+\epsilon\sqrt{H(Z_t^{i,N})}+\epsilon\sqrt{H(\bar{Z}_t^i)}\right)\left(\tilde{H}\left(\bar{Z}_t^j\right)+\tilde{H}(Z_t^{j,N})\right)\\
\leq&\mathcal{C}_z\sum_{i,j=1}^{N}f(r_t^i)\left(\tilde{H}\left(\bar{Z}_t^j\right)+\tilde{H}(Z_t^{j,N})\right)\\
&+\epsilon\mathcal{C}_z\sum_{i,j=1}^{N}f(r_t^i)\left(\sqrt{H(Z_t^{i,N})}+\sqrt{H(\bar{Z}_t^i)}\right)\left(\tilde{H}\left(\bar{Z}_t^j\right)+\tilde{H}(Z_t^{j,N})\right).
\end{aligned}
$$

Using (4.2.10) from Lemma 4.2.6, we obtain for the first sum:

$$
\begin{aligned}
&\mathcal{C}_z\sum_{i,j=1}^{N}f(r_t^i)\left(\tilde{H}\left(\bar{Z}_t^j\right)+\tilde{H}(Z_t^{j,N})\right)\\
&\leq \mathcal{C}_z\sum_{i,j=1}^{N}f(r_t^i)\left(H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)+H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right).
\end{aligned}
$$

With (4.2.11) from the same Lemma, we obtain for the second sum:

$$
\begin{aligned}
&\epsilon\mathcal{C}_z\sum_{i,j=1}^{N}f(r_t^i)\left(\sqrt{H(Z_t^{i,N})}+\sqrt{H(\bar{Z}_t^i)}\right)\left(\tilde{H}\left(\bar{Z}_t^j\right)+\tilde{H}(Z_t^{j,N})\right)\\
\leq&\epsilon\mathcal{C}_z\frac{2}{a}\sum_{i,j=1}^{N}f(r_t^i)\left(\sqrt{H(Z_t^{i,N})}+\sqrt{H(\bar{Z}_t^i)}\right)\left[\sqrt{H\left(\bar{Z}_t^j\right)}\exp\left(a\sqrt{H\left(\bar{Z}_t^j\right)}\right)\right.\\
&\qquad\left.+\sqrt{H\left(Z_t^{j,N}\right)}\exp\left(a\sqrt{H\left(Z_t^{j,N}\right)}\right)\right].
\end{aligned}
$$

Since for all $(y_1, y_2, y_3, y_4) \in (\mathbb{R}^+)^4$, we have

$$(y_1 + y_3)\left(y_3 e^{a y_3} + y_4 e^{a y_4}\right) \leq 2\left(y_1^2 e^{a y_1} + y_3^2 e^{a y_3} + y_3^2 e^{a y_3} + y_4^2 e^{a y_4}\right),$$

we obtain for this last sum

$$\begin{aligned}
&\frac{2\epsilon \mathcal{C}_z}{a} \sum_{i,j=1}^{N} f(r_t^i) \left(\sqrt{H(Z_t^{i,N})} + \sqrt{H(\bar{Z}_t^i)}\right) \Bigg[\sqrt{H\left(\bar{Z}_t^j\right)} \exp\left(a\sqrt{H\left(\bar{Z}_t^j\right)}\right) \\
&\qquad\qquad + \sqrt{H\left(Z_t^{j,N}\right)} \exp\left(a\sqrt{H\left(Z_t^{j,N}\right)}\right)\Bigg] \\
&\leq \frac{4\epsilon \mathcal{C}_z}{a} \sum_{i,j=1}^{N} f(r_t^i) \left(H(\bar{Z}_t^i) \exp\left(a\sqrt{H(\bar{Z}_t^i)}\right) + H(Z_t^{i,N}) \exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right) \\
&\quad + \frac{4\epsilon \mathcal{C}_z}{a} \sum_{i,j=1}^{N} f(r_t^i) \left(H(\bar{Z}_t^j) \exp\left(a\sqrt{H(\bar{Z}_t^j)}\right) + H(Z_t^{j,N}) \exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right) \\
&\leq \frac{4\epsilon \mathcal{C}_z}{a} N \sum_{i=1}^{N} f(r_t^i) \left(H(\bar{Z}_t^i) \exp\left(a\sqrt{H(\bar{Z}_t^i)}\right) + H(Z_t^{i,N}) \exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right) \\
&\quad + \frac{4\epsilon \mathcal{C}_z}{a} \sum_{i,j=1}^{N} f(r_t^i) \left(H(\bar{Z}_t^j) \exp\left(a\sqrt{H(\bar{Z}_t^j)}\right) + H(Z_t^{j,N}) \exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right).
\end{aligned}$$

Then, by reconsidering the first expression:

$$\begin{aligned}
&\frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{N} f'\left(r_t^i\right) G_t^i \sum_{j=1}^{N} \|Z_t^{j,N} - \bar{Z}_t^j\|_1\right) \\
&\leq \frac{\mathcal{C}_1}{N} \sum_{i=1}^{N} f(r_t^i) G_t^i \\
&\quad + \frac{2\epsilon}{N^2} \mathcal{C}_z \sum_{i,j=1}^{N} f(r_t^i) \left(H(\bar{Z}_t^j) \exp\left(a\sqrt{H(\bar{Z}_t^j)}\right) + H(Z_t^{j,N}) \exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right) \\
&\quad + \frac{2\epsilon}{N^2} \frac{4\epsilon \mathcal{C}_z}{a} N \sum_{i=1}^{N} f(r_t^i) \left(H(\bar{Z}_t^i) \exp\left(a\sqrt{H(\bar{Z}_t^i)}\right) + H(Z_t^{i,N}) \exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right) \\
&\quad + \frac{2\epsilon}{N^2} \frac{4\epsilon \mathcal{C}_z}{a} \sum_{i,j=1}^{N} f(r_t^i) \left(H(\bar{Z}_t^j) \exp\left(a\sqrt{H(\bar{Z}_t^j)}\right) + H(Z_t^{j,N}) \exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right)
\end{aligned}$$

This way, by Assumption 4.2.8 since

$$L_X \mathcal{C}_1 \leq \frac{c}{2}, \quad 2\mathcal{C}_z L_X \leq \frac{\lambda}{64} \quad \text{and} \quad L_X \epsilon \frac{8\mathcal{C}_z}{a} \leq \frac{\lambda}{64},$$

we get

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{N}f'\left(r_t^i\right)G_t^i\sum_{j=1}^{N}\|Z_t^{j,N}-\bar{Z}_t^j\|_1\right)\\
\leq&\frac{1}{N}\frac{c}{2L_X}\sum_{i=1}^{N}f(r_t^i)G_t^i\\
&+\frac{\epsilon}{N^2}\frac{\lambda}{64L_X}\sum_{i,j=1}^{N}f(r_t^i)\left(H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)+H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right)\\
&+\frac{\epsilon}{N}\frac{\lambda}{64L_X}\sum_{i=1}^{N}f(r_t^i)\left(H(\bar{Z}_t^i)\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)+H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right)\\
&+\frac{\epsilon}{N^2}\frac{\lambda}{64L_X}\sum_{i,j=1}^{N}f(r_t^i)\left(H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)+H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right)
\end{aligned}$$

and we finally obtain "half" the result

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}G_t^if'(r_t^i)\left(\frac{L_X}{N}\left(\sum_{j=1}^{N}\|Z_t^{j,N}-\bar{Z}_t^j\|_1\right)\right)-\frac{c}{2}\frac{1}{N}\sum_{i=1}^{N}f(r_t^i)G_t^i\\
&-\frac{\epsilon}{2}\frac{1}{N}\sum_{i=1}^{N}f(r_t^i)\left[\frac{\lambda}{16}H(\bar{Z}_t^i)\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)+\frac{\lambda}{16}H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right.\\
&\qquad\left.+\frac{\lambda}{16N}\sum_{j=1}^{N}H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)+\frac{\lambda}{16N}\sum_{j=1}^{N}H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right]\leq 0.
\end{aligned}$$

Likewise, by Assumption 4.2.8, since

$$\delta L_C\mathcal{C}_1\leq\frac{c}{2},\qquad 2\mathcal{C}_z\delta L_C\leq\frac{\lambda}{64}\qquad\text{and}\qquad \delta L_C\epsilon\frac{8\mathcal{C}_z}{a}\leq\frac{\lambda}{64},$$

we obtain the second "half"

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\delta G_t^if'(r_t^i)\left(\frac{L_C}{N}\left(\sum_{j=1}^{N}\|Z_t^{j,N}-\bar{Z}_t^j\|_1\right)\right)-\frac{c}{2}\frac{1}{N}\sum_{i=1}^{N}f(r_t^i)G_t^i\\
&-\frac{\epsilon}{2}\frac{1}{N}\sum_{i=1}^{N}f(r_t^i)\left[\frac{\lambda}{16}H(\bar{Z}_t^i)\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)+\frac{\lambda}{16}H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right.\\
&\left.\frac{\lambda}{16N}\sum_{j=1}^{N}H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)+\frac{\lambda}{16N}\sum_{j=1}^{N}H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right]\leq 0.
\end{aligned}$$

Eventually, we have proved $\sum_{i=1}^{N}I_t^{2,i}\leq 0$. $\square$

Proof of Lemma 4.3.4. Since $f'(r) \leq 1$, we have by Cauchy-Schwarz

$$\begin{aligned}
&\mathbb{E}\left(G_t^i f'(r_t^i)\left(\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|\right)\right)\\
\leq &\mathbb{E}\left(|G_t^i|^2\right)^{1/2}\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|^2\right)^{1/2}.
\end{aligned}$$

By Lemma 4.3.2, we have for each $i \leq N$, for all $t \geq 0$, $\mathbb{E}[(G_t^i)^2] \leq \mathcal{C}_{G,2}$.

Moreover, we notice that $(\bar{Z}_t^j)_j$ are i.i.d with law $\bar{\mu}_t$. Let's denote $\bar{Z}_t$ a generic random variable of law $\bar{\mu}_t$ independent of $\bar{Z}_t^i$. The calculus of the right term of the product has already be done in Subsection 4.1.4, and we have (4.1.9):

$$\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|^2\right) \leq \frac{8L_X^2}{N}\mathbb{E}(\|\bar{Z}_t\|_1^2) \leq \frac{8L_X^2}{N-1}\mathbb{E}(\|\bar{Z}_t\|_1^2).$$

A similar calculus gives us

$$\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^N K_C(\bar{Z}_t^i - \bar{Z}_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\right|^2\right) \leq \frac{8L_C^2}{N-1}\mathbb{E}(\|\bar{Z}_t\|_1^2).$$

By Lemma 4.2.5, $\mathbb{E}\left(|\bar{X}_t|^2 + |\bar{C}_t|^2\right) \leq C_0$. In particular,

$$\mathbb{E}\left(\|\bar{Z}_t\|_1^2\right) = \mathbb{E}\left(\left[|\bar{X}_t| + |\bar{C}_t|\right]^2\right) \leq 2\mathbb{E}\left(|\bar{X}_t|^2 + |\bar{C}_t|^2\right) \leq 2C_0.$$

Thus

$$\mathbb{E}\left(G_t^i f'(r_t^i)\left(\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|\right)\right) \leq L_X \mathcal{C}_{G,2}^{1/2}\sqrt{2C_0}\sqrt{\frac{8}{N-1}}$$

and likewise

$$\mathbb{E}\left(G_t^i f'(r_t^i)\left(\left|\frac{1}{N}\sum_{j=1}^N K_C(\bar{Z}_t^i - \bar{Z}_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\right|\right)\right) \leq L_C \mathcal{C}_{G,2}^{1/2}\sqrt{2C_0}\sqrt{\frac{8}{N-1}}.$$

□

4.3.4 Contraction in various regions of space

The goal of this section is to prove the Lemma 4.3.3: for each $i \leq N$, for all $t > 0$,

$$\mathbb{E}\tilde{K}_t^i \leq \xi\left(2 + \delta\gamma + L_X + \delta L_C - L_C - \frac{1 + L_X}{\delta}\right)\mathbb{E}G_t^i.$$

Recall

$$\begin{aligned}\tilde{K}^i_t =& G^i_t\Big[2cf(r^i_t) + f''(r^i_t)\left(2\sigma_x^2\varphi_{\mathrm{rc}}\left(|X^{i,N}_t - \bar{X}^i_t|\right)^2\right)\\ &+ f'(r^i_t)\Big((1+\gamma\delta + L_X + \delta L_C)|X^{i,N}_t - \bar{X}^i_t| - |(X^{i,N}_t)^3 - (\bar{X}^i_t)^3|\\ &+ (1+L_X+\delta L_C - \delta)|C^{i,N}_t - \bar{C}^i_t| + \left(\epsilon\mathcal{C}_1^{(f)} + \mathcal{C}_2^{(f)}\right)\sigma_x^2\varphi_{\mathrm{rc}}\left(|X^{i,N}_t - \bar{X}^i_t|\right)^2 r^i_t\Big)\Big]\\ &+ \epsilon f(r^i_t)\left(4\tilde{B} - \frac{\lambda}{16}\tilde{H}(\bar{Z}^i_t) - \frac{\lambda}{16}\tilde{H}(Z^{i,N}_t) - \frac{\lambda}{16N}\sum_{j=1}^N \tilde{H}(\bar{Z}^j_t) - \frac{\lambda}{16N}\sum_{j=1}^N \tilde{H}(Z^{j,N}_t)\right),\end{aligned}$$

which is a quantity that contains every term we have not yet dealt with. To prove Lemma 4.3.3, we divide for each $i \in \{1,...,N\}$ the space into three regions

$$\begin{aligned}\mathrm{Reg}^i_1 &= \left\{(\bar{Z}^i_t, Z^{i,N}_t) \text{ s.t. } r^i_t \le R \text{ and } |\bar{X}^i_t - X^{i,N}_t| \ge \xi\right\},\\ \mathrm{Reg}^i_2 &= \left\{(\bar{Z}^i_t, Z^{i,N}_t) \text{ s.t. } r^i_t \le R \text{ and } |\bar{X}^i_t - X^{i,N}_t| < \xi\right\},\\ \mathrm{Reg}^i_3 &= \left\{(\bar{Z}^i_t, Z^{i,N}_t) \text{ s.t. } r^i_t > R\right\},\end{aligned}$$

and consider

$$\frac{1}{N}\sum_{i=1}^N \mathbb{E}\tilde{K}^i_t = \frac{1}{N}\sum_{i=1}^N\left(\mathbb{E}\left(\tilde{K}^i_t\mathbb{1}_{\mathrm{Reg}^i_1}\right) + \mathbb{E}\left(\tilde{K}^i_t\mathbb{1}_{\mathrm{Reg}^i_2}\right) + \mathbb{E}\left(\tilde{K}^i_t\mathbb{1}_{\mathrm{Reg}^i_3}\right)\right).$$

This division in different regions isolates the region 1, where the coupling is reflexive. Nevertheless, it doesn't mean the coupling is antithetic on the two others. The function f is constant on region 3, therefore terms in φ_{rc} will be null. In region 2, the choice of the function f will make this term non-positive.

In fact, we will prove that for each $i \le N$, for all $t > 0$

$$\begin{aligned}&\tilde{K}^i_t\mathbb{1}_{\mathrm{Reg}^i_1} \le 0 \quad \text{ and } \quad \tilde{K}^i_t\mathbb{1}_{\mathrm{Reg}^i_3} \le 0,\\ &\mathbb{E}\left(\tilde{K}^i_t\mathbb{1}_{\mathrm{Reg}^i_2}\right) \le \xi\left(2 + \delta\gamma + L_X + \delta L_C - L_C - \frac{1+L_X}{\delta}\right)\mathbb{E}\left(G^i_t\right).\end{aligned}$$

4.3.4.1 Region 1 : $\xi \le |X^{i,N}_t - \bar{X}^i_t|$ and $r^i_t \le R$.

In this region of space, since the coupling is reflexive and $\varphi_{\mathrm{rc}}(|X^{i,N}_t - \bar{X}^i_t|) = 1$, we have

$$\begin{aligned}\tilde{K}^i_t\mathbb{1}_{\mathrm{Reg}^i_1} =& G^i_t\Big[2cf(r^i_t) + 2\sigma_x^2 f''(r^i_t) + f'(r^i_t)\left(\epsilon\mathcal{C}_1^{(f)} + \mathcal{C}_2^{(f)}\right)\sigma_x^2 r^i_t\\ &+ f'(r^i_t)(1+\gamma\delta + L_X + \delta L_C)|X^{i,N}_t - \bar{X}^i_t|\Big] - G^i_t f'(r^i_t)|(X^{i,N}_t)^3 - (\bar{X}^i_t)^3|\\ &- G^i_t f'(r^i_t)(\delta - 1 - L_X - \delta L_C)|C^{i,N}_t - \bar{C}^i_t| + \epsilon f(r^i_t)4\tilde{B}\\ &- \epsilon f(r^i_t)\left(\frac{\lambda}{16}\tilde{H}(\bar{Z}^i_t) + \frac{\lambda}{16}\tilde{H}(Z^{i,N}_t) + \frac{\lambda}{16N}\sum_{j=1}^N \tilde{H}(\bar{Z}^j_t) + \frac{\lambda}{16N}\sum_{j=1}^N \tilde{H}(Z^{j,N}_t)\right),\end{aligned}$$

and since $\tilde{H}(z) \geq 0$, $|X_t^{i,N} - \bar{X}_t^i| \leq r_t^i$, $\delta > \frac{1+L_X}{1-L_C}$ (by Assumption 4.2.8(iii)) and $1 \leq G_t^i$ we have

$$\begin{aligned}\tilde{K}_t^i \mathbb{1}_{\mathrm{Reg}_1^i} \leq & G_t^i \Big[(2c+4\epsilon\tilde{B})f(r_t^i) + 2\sigma_x^2 f''(r_t^i) \\ & + f'(r_t^i)\left(1+\delta\gamma+L_X+\delta L_C+\left(\epsilon\mathcal{C}_1^{(f)}+\mathcal{C}_2^{(f)}\right)\sigma_x^2\right)r_t^i\Big].\end{aligned}$$

Using the definition f given in (4.2.22) we get

$$\begin{aligned}& 2\sigma_x^2 f''(r_t^i) + f'(r_t^i)\left(1+\delta\gamma+L_X+L_C+\left(\epsilon\mathcal{C}_1^{(f)}+\mathcal{C}_2^{(f)}\right)\sigma_x^2\right)r_t^i \\ =& 2\sigma_x^2\phi'(r_t^i)g(r_t^i) + 2\sigma_x^2\phi(r_t^i)g'(r_t^i) \\ & + \phi(r_t^i)g(r_t^i)\left(1+\delta\gamma+L_X+\delta L_C+\left(\epsilon\mathcal{C}_1^{(f)}+\mathcal{C}_2^{(f)}\right)\sigma_x^2\right)r_t^i \\ =& 2\sigma_x^2\phi(r_t^i)g'(r_t^i) \\ =& -(2c+4\epsilon\tilde{B})\Phi(r_t^i).\end{aligned}$$

Thus

$$\begin{aligned}(2c+4\epsilon\tilde{B})f(r_t^i) + 2\sigma_x^2 f''(r_t^i) + f'(r_t^i)\left(1+\delta\gamma+L_X+\delta L_C+\left(\epsilon\mathcal{C}_1^{(f)}+\mathcal{C}_2^{(f)}\right)\sigma_x^2\right)r_t^i & \\ = (2c+4\epsilon\tilde{B})f(r_t^i) - (2c+4\epsilon\tilde{B})\Phi(r_t^i) & \quad (4.3.12) \\ \leq 0. &\end{aligned}$$

Eventually, in this region of space

$$\tilde{K}_t^i \mathbb{1}_{\mathrm{Reg}_1^i} \leq 0.$$

4.3.4.2 Region 2 : $|X_t^{i,N} - \bar{X}_t^i| < \xi$ and $r_t^i \leq R$.

In this region, we can write $\tilde{K}_t^i$ as

$$\begin{aligned}\tilde{K}_t^i \mathbb{1}_{\mathrm{Reg}_2^i} =& G_t^i\Big[2cf(r_t^i) + \varphi_{\mathrm{rc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)^2\left[2\sigma_x^2 f''(r_t^i) + \left(\epsilon\mathcal{C}_1^{(f)}+\mathcal{C}_2^{(f)}\right)\sigma_x^2 r_t^i f'(r_t^i)\right] \\ & + f'(r_t^i)\Big((1+\gamma\delta+L_X+\delta L_C)|X_t^{i,N}-\bar{X}_t^i| \\ & -(\delta-1-L_X-\delta L_C)|C_t^{i,N}-\bar{C}_t^i|\Big)\Big] - G_t^i f'(r_t^i)|(X_t^{i,N})^3-(\bar{X}_t^i)^3| + \epsilon f(r_t^i)4\tilde{B} \\ & - \epsilon f(r_t^i)\left(\frac{\lambda}{16}\tilde{H}(\bar{Z}_t^i) + \frac{\lambda}{16}\tilde{H}(Z_t^{i,N}) + \frac{\lambda}{16N}\sum_{j=1}^N \tilde{H}(\bar{Z}_t^j) + \frac{\lambda}{16N}\sum_{j=1}^N \tilde{H}(Z_t^{j,N})\right).\end{aligned}$$

Since $r_t^i = |X_t^{i,N} - \bar{X}_t^i| + \delta|C_t^{i,N} - \bar{C}_t^i|$ and $|X_t^{i,N} - \bar{X}_t^i| < \xi$, we have $|C_t^{i,N} - \bar{C}_t^i| \geq (r_t^i - \xi)/\delta$. Since $\delta > \dfrac{1+L_X}{1-L_C}$, we obtain

$$\begin{aligned}
\tilde{K}_t^i \mathbb{1}_{\mathrm{Reg}_2^i} \leq & G_t^i \Big[2cf(r_t^i) + \varphi_{\mathrm{rc}} \left(|X_t^{i,N} - \bar{X}_t^i| \right)^2 \left[2\sigma_x^2 f''(r_t^i) + \left(\epsilon \mathcal{C}_1^{(f)} + \mathcal{C}_2^{(f)} \right) \sigma_x^2 r_t^i f'(r_t^i) \right] \\
& + f'(r_t^i) \left((1 + \gamma\delta + L_X + \delta L_C)\xi - (\delta - \delta L_C - 1 - L_X) \frac{r_t^i - \xi}{\delta} \right) \Big] \\
& + \epsilon f(r_t^i) 4\tilde{B} \\
\leq & \varphi_{\mathrm{rc}} \left(|X_t^{i,N} - \bar{X}_t^i| \right)^2 G_t^i \left[2\sigma_x^2 f''(r_t^i) + \left(\epsilon \mathcal{C}_1^{(f)} + \mathcal{C}_2^{(f)} \right) \sigma_x^2 r_t^i f'(r_t^i) \right] \\
& + G_t^i f'(r_t^i) \xi \left[1 + \gamma\delta + L_X + \delta L_C + 1 - L_C - \frac{1+L_X}{\delta} \right] \\
& + G_t^i \left((2c + 4\epsilon\tilde{B}) f(r_t^i) - r_t^i f'(r_t^i) \left(1 - L_C - \frac{1+L_X}{\delta} \right) \right).
\end{aligned}$$

By (4.3.12),

$$\begin{aligned}
& 2\sigma_x^2 f''(r_t^i) + \left(\epsilon \mathcal{C}_1^{(f)} + \mathcal{C}_2^{(f)} \right) \sigma_x^2 r_t^i f'(r_t^i) \\
& \qquad = -(2c + 4\epsilon\tilde{B})\Phi(r_t^i) - f'(r_t^i) r_t^i (1 + \delta\gamma + L_X + L_C) \leq 0,
\end{aligned}$$

and by Lemma 4.2.9

$$2c + 4\epsilon\tilde{B} \leq \left(1 - L_C - \frac{1+L_X}{\delta} \right) \min_{r \in]0,R]} \frac{f'(r)r}{f(r)},$$

we obtain

$$\tilde{K}_t^i \mathbb{1}_{\mathrm{Reg}_2^i} \leq G_t^i f'(r_t^i) \xi \left[1 + \gamma\delta + L_X + \delta L_C + 1 - L_C - \frac{1+L_X}{\delta} \right].$$

Finally, since $f'(r) \leq 1$,

$$\mathbb{E}\tilde{K}_t^i \mathbb{1}_{\mathrm{Reg}_2^i} \leq \xi \left(2 + \delta\gamma + L_X + \delta L_C - L_C - \frac{1+L_X}{\delta} \right) \mathbb{E}G_t^i.$$

4.3.4.3 Region 3 : $r_t^i \geq R$.

In this region of space $f' = f'' = 0$ and f is constant, and we therefore have

$$\begin{aligned}
\tilde{K}_t^i \mathbb{1}_{\mathrm{Reg}_3^i} = & f(r_t^i) \Bigg[2cG_t^i + 4\epsilon\tilde{B} - \frac{\lambda\epsilon}{16} \Bigg(\tilde{H}(\bar{Z}_t^i) + \tilde{H}(Z_t^{i,N}) + \frac{1}{N} \sum_{j=1}^N \tilde{H}(\bar{Z}_t^j) \\
& + \frac{1}{N} \sum_{j=1}^N \tilde{H}(Z_t^{j,N}) \Bigg) \Bigg].
\end{aligned}$$

Since $G_t^i = 1+\epsilon\tilde{H}(\bar{Z}_t^i)+\epsilon\tilde{H}(Z_t^{i,N})+\frac{\epsilon}{N}\sum_{j=1}^N \tilde{H}(Z_t^{j,N})+\frac{\epsilon}{N}\sum_{j=1}^N \tilde{H}(\bar{Z}_t^j)$ by definition (4.2.27), we can write

$$\tilde{K}_t^i \mathbb{1}_{\mathrm{Reg}_3^i} = f(r_t^i)\left[2c+4\epsilon\tilde{B}\left(2c-\frac{\lambda\epsilon}{16}\right)\left(\tilde{H}(\bar{Z}_t^i)+\tilde{H}(Z_t^{i,N})+\frac{1}{N}\sum_{j=1}^N \tilde{H}(\bar{Z}_t^j) + \frac{1}{N}\sum_{j=1}^N \tilde{H}(Z_t^{j,N})\right)\right]$$

Since $c \leq \lambda/32$ from Assumption 4.2.8, we obtain

$$\tilde{K}_t^i \mathbb{1}_{\mathrm{Reg}_3^i} \leq f(r_t^i)\left[2c+4\epsilon\tilde{B}-\epsilon\left(\frac{\lambda}{16}-2c\right)(\tilde{H}(\bar{Z}_t^i)+\tilde{H}(Z_t^{i,N}))\right].$$

We have chosen R such that, for z,z' satisfying $r \geq R$, we have $H(z)+H(z') \geq \frac{64}{15}\frac{B}{\lambda}$ by Lemma 4.2.2 (iv). Therefore

$$\begin{aligned}\tilde{K}_t^i \mathbb{1}_{\mathrm{Reg}_3^i} &\leq f(r_t^i)\left(2c+4\epsilon\tilde{B}-\epsilon(\lambda-2c)\frac{64}{15}\frac{\tilde{B}}{\lambda}\right)\\ &= f(r_t^i)\left(c\left(2+\frac{128}{15}\frac{\epsilon\tilde{B}}{\lambda}\right)-\frac{4}{15}\epsilon\tilde{B}\right)\end{aligned}$$

Assumptions 4.2.8 and more specifically the inequality

$$c \leq \frac{\lambda}{32}\frac{\frac{64\epsilon\tilde{B}}{15\lambda}}{1+\frac{64\epsilon\tilde{B}}{15\lambda}} = \frac{\frac{4}{15}\epsilon\tilde{B}}{2+\frac{128}{15}\frac{\epsilon\tilde{B}}{\lambda}}$$

yields the desired result: $\tilde{K}_t^i \mathbb{1}_{\mathrm{Reg}_3^i} \leq 0$.

4.A Various technical lemmas

4.A.1 On Lemma 4.2.2

Lemma 4.A.1. For all $z,z' \in \mathbb{R}^d$, denoting $r(z,z') = |x-x'|+\delta|c-c'|$

$$r(z,z')^2 \leq \frac{16(1+\delta^2)}{\min(\gamma,1)}\left(H(z)+H(z')\right), \tag{4.A.1}$$

so that, in particular, for any constant $B>0$, if $r(z,z') \geq R = \sqrt{\frac{1024(1+\delta^2)B}{15\lambda\min(\gamma,1)}}$, then

$$\lambda H(z)+\lambda H(z') \geq \frac{64}{15}B.$$

Proof. We have $H(z) \geq \frac{\gamma}{4}x^2 + \frac{c^2}{4} \geq \frac{1}{4}\min(\gamma, 1)\left(x^2 + c^2\right)$. Thus

$$\begin{aligned}
r(z,z')^2 =& \left(|x-x'| + \delta|c-c'|\right)^2 \\
\leq& 2|x-x'|^2 + 2\delta^2|c-c'|^2 \\
\leq& 4x^2 + 4x'^2 + 4\delta^2 c^2 + 4\delta^2 c'^2 \\
\leq& 4(1+\delta^2)(x^2+c^2) + 4(1+\delta^2)(x'^2+c'^2) \\
\leq& 16\frac{(1+\delta^2)}{\min(\gamma,1)}\left(H(z)+H(z')\right)
\end{aligned}$$

□

4.A.2 Proof of Lyapunov's property of H and its consequences

Lyapunov's property

Proof of Lemma 4.2.3. First, we prove (4.2.4). It will yield (4.2.5) as we describe it at the end. We notice

$$\partial_c H = c + \alpha \quad \text{and} \quad \partial_x H = \gamma x + \beta,$$

so

$$\begin{aligned}
\mathcal{L}_\mu H(z) =& \partial_x H(z)(x - x^3) + \partial_x H(z) K_X * \mu(z) - c\partial_c H(z) + \partial_c H(z) K_C * \mu(z) \\
&+ \frac{\sigma_x^2 \gamma}{2} + \frac{\sigma_c^2}{2} \\
=& (\gamma x + \beta)(x - x^3) - c(c+\alpha) + (\gamma x + \beta) K_X * \mu(z) + (c+\alpha) K_C * \mu(z) \\
&+ \frac{\sigma_x^2 \gamma}{2} + \frac{\sigma_c^2}{2}.
\end{aligned}$$

First, we focus on interaction terms. We have

$$\begin{aligned}
|K_X * \mu(z)| \leq& \int_{\mathbb{R}^2} |K_X(z - z')|\mu(dz') \\
=& \int_{\mathbb{R}^2} L_X(\|z\|_1 + \|z'\|_1)\mu(dz').
\end{aligned}$$

Hence,

$$\begin{aligned}
(\gamma x + \beta) K_X * \mu(z) \leq& L_X(\gamma|x| + \beta)(|x| + |c| + \mathbb{E}_\mu(|X|) + \mathbb{E}_\mu(|C|)) \\
\leq& L_X \left[\gamma|x|^2 + \gamma|x||c| + \gamma|x|\mathbb{E}_\mu(|X|) + \gamma|x|\mathbb{E}_\mu(|C|) + \beta|x| + \beta|c| \right.\\
&\left. + \beta\mathbb{E}_\mu(|X|) + \beta\mathbb{E}_\mu(|C|)\right],
\end{aligned}$$

and using Young's inequality $ab \leq \frac{\alpha}{2}a^2 + \frac{1}{2\alpha}b^2$ ($\alpha = 16$ when we separate x term and c term, and $\alpha = 1$ otherwise on the various terms we get

$$\begin{aligned}
(\gamma x + \beta) K_X * \mu(z) \leq& L_X \Big(\gamma|x|^2 + 8\gamma^2|x|^2 + \frac{|c|^2}{32} + \frac{\gamma}{2}|x|^2 + \frac{\gamma}{2}\mathbb{E}_\mu(|X|)^2 + 8\gamma^2|x|^2 \\
&+ \frac{\mathbb{E}_\mu(|C|)^2}{32} + \frac{\beta^2}{2} + \frac{|x|^2}{2} + 8\beta^2 + \frac{|c|^2}{32} + \frac{\beta^2}{2} + \frac{1}{2}\mathbb{E}_\mu(|X|)^2 \\
&+ 8\beta^2 + \frac{\mathbb{E}_\mu(|C|)^2}{32}\Big),
\end{aligned}$$

so

$$
\begin{aligned}
(\gamma x+\beta)K_X*\mu(z) \le & L_X\left(17\beta^2+|x|^2\left(\frac{1}{2}+\frac{3}{2}\gamma+16\gamma^2\right)+\frac{|c|^2}{16}\right. \\
& \left. +\mathbb{E}_\mu(|X|)^2\left(\frac{\gamma}{2}+\frac{1}{2}\right)+\frac{\mathbb{E}_\mu(|C|)^2}{16}\right).
\end{aligned}
$$

Likewise

$$
\begin{aligned}
(c+\alpha)K_C*\mu(z) \le & L_C\left(17\alpha^2+\frac{17}{2}|x|^2+|c|^2\left(\frac{3}{2}+\frac{1}{16}\right)+\mathbb{E}_\mu(|X|)^2\frac{17}{2}\right. \\
& \left. +\mathbb{E}_\mu(|C|)^2\left(\frac{1}{2}+\frac{1}{32}\right)\right).
\end{aligned}
$$

The idea is then to bound $\lambda H(z)+\mathcal{L}_\mu H(z)$, by distinguishing 3 types of terms: we isolate terms in $\mathbb{E}_\mu(|C|)^2-c^2$, $\mathbb{E}_\mu(|X|)^2-x^2$, and we group polynomials terms. Then, we notice the polynomial is upper bounded by a constant A. Thus

$$
\begin{aligned}
&\lambda H(z)+\mathcal{L}_\mu H(z)-\frac{\sigma_x^2\gamma}{2}-\frac{\sigma_c^2}{2} \\
&\quad =\lambda\left(\frac{1}{2}\gamma x^2+\beta x+\frac{1}{2}c^2+\alpha c+H_0\right)+(\gamma x+\beta)(x-x^3)-c(c+\alpha) \\
&\qquad +(\gamma x+\beta)K_X*\mu(z)+(c+\alpha)K_C*\mu(z) \\
&\quad \le \left(\lambda H_0+17\beta^2L_X+17\alpha^2L_C\right)-\gamma x^4-\beta x^3+(1+\lambda)\beta x \\
&\qquad +\left(\left(1+\frac{\lambda}{2}\right)\gamma+L_X\left(\frac{1}{2}+\frac{3}{2}\gamma+16\gamma^2\right)+\frac{17}{2}L_C+L_X\left(\frac{\gamma}{2}+\frac{1}{2}\right)+\frac{17}{2}L_C\right)x^2 \\
&\qquad +\left(\frac{L_X}{16}+L_C\left(\frac{3}{2}+\frac{1}{16}\right)+\frac{L_X}{16}+L_C\left(\frac{1}{2}+\frac{1}{32}\right)-\left(1-\frac{\lambda}{2}\right)\right)c^2-(1-\lambda)\alpha c \\
&\qquad +\left(\frac{L_X}{16}+\frac{L_C}{2}+\frac{L_C}{32}\right)\left(\mathbb{E}_\mu(|C|)^2-c^2\right) \\
&\qquad +\left(\frac{\gamma}{2}L_X+\frac{1}{2}L_X+\frac{17}{2}L_C\right)\left(\mathbb{E}_\mu(|X|)^2-x^2\right)
\end{aligned}
$$

According Assumption 4.1.1 and Remark 4.2.7

$$
\frac{L_X}{8}+L_C\left(2+\frac{3}{32}\right)<1-\frac{\lambda}{2},
$$

the coefficient of c^2 is negative, and there exists $A\ge 0$ such that

$$
\begin{aligned}
-\gamma x^4 & -\beta x^3+(1+\lambda)\beta x \\
&+\left((1+\frac{\lambda}{2})\gamma+\frac{L_X}{2}+\frac{3L_X}{2}\gamma+16\gamma^2L_X+\frac{17}{2}L_C+\frac{\gamma}{2}L_X+\frac{1}{2}L_X+\frac{17}{2}L_C\right)x^2 \\
&+\left(\frac{L_X}{16}+\frac{3L_C}{2}+\frac{L_C}{16}+\frac{L_X}{16}+\frac{L_C}{2}+\frac{L_C}{32}-\left(1-\frac{\lambda}{2}\right)\right)c^2-(1-\lambda)\alpha c\le A.
\end{aligned}
$$

Hence the result:

$$\begin{aligned}\mathcal{L}_\mu H(\bar z) \leq& B + (\alpha_X L_X + \beta_X L_C)\left(\mathbb{E}_\mu(|X|)^2 - \bar x^2\right) + (\alpha_C L_X + \beta_C L_C)\left(\mathbb{E}_\mu(|C|)^2 - \bar c^2\right)\\ &- \lambda H(\bar z).\end{aligned}$$

This inequation yields to (4.2.5) by applying the link between $\mathcal{L}_{\mu_{\mathrm{emp}}}$ and $\mathcal{L}^{i,N}\bar h^i(z_1,\dots,z_N)$ described in (4.2.1).
Let's have $(z_i)_{1\leq i\leq N} \in \mathbb{R}^{2N}$ and denote $H:(z_1,\dots,z_N)\mapsto H(z_i)$. We define

$$\mu_{\mathrm{emp}} = \frac{1}{N}\sum_j \delta_{z_j}.$$

Then

$$\begin{aligned}&\mathcal{L}^{i,N}H(z_1,\dots,z_N) = \mathcal{L}_{\mu_{\mathrm{emp}}}H(z_i)\\ &\quad\leq B + (\alpha_X L_X + \beta_X L_C)\left(\mathbb{E}_{\mu_{\mathrm{emp}}}(|X|)^2 - \bar x^2\right) + (\alpha_C L_X + \beta_C L_C)\left(\mathbb{E}_{\mu_{\mathrm{emp}}}(|C|)^2 - \bar c^2\right)\\ &\qquad - \lambda H(\bar z).\end{aligned}$$

Since

$$\mathbb{E}_{\mu_{\mathrm{emp}}}(|X|) = \frac{1}{N}\sum_{j=1}^N |x_j| \quad \text{and} \quad \mathbb{E}_{\mu_{\mathrm{emp}}}(|C|) = \frac{1}{N}\sum_{j=1}^N |c_j|,$$

we obtain (4.2.5). □

First consequences

Proof of Proposition 4.2.4. The first inequality (4.2.6) relies on Dykin's formula: for all $t\geq 0$,

$$\mathbb{E}H\left(\bar Z_t^i\right) = \mathbb{E}H(\bar Z_0^i) + \mathbb{E}\left[\int_0^t \mathcal{L}_\mu H(\bar Z_s)ds\right].$$

Then, we can differentiate this expression. Since $(\omega,t)\to \left|\mathcal{L}_\mu H(\bar X_t,\bar C_t)\right|$ is bounded by an integrable function of ω

$$\begin{aligned}\frac{d}{dt}\mathbb{E}H\left(\bar Z_t^i\right) =& \mathbb{E}\left[\mathcal{L}_\mu H(\bar Z_t)\right]\\ \leq& B + (\alpha_X L_X + \beta_X L_C)\,\mathbb{E}\left[\left(\mathbb{E}_\mu(|X|)^2 - \bar X_t^2\right)\right]\\ &+ (\alpha_C L_X + \beta_C L_C)\,\mathbb{E}\left[\left(\mathbb{E}_\mu(|C|)^2 - \bar C_t^2\right)\right] - \lambda\mathbb{E}\left[H(\bar Z_t)\right]\\ \leq& B - \lambda\mathbb{E}\left[H(\bar X_t,\bar C_t)\right] - (\alpha_X L_X + \beta_X L_C)\operatorname{Var}(\bar X_t)\\ &- (\alpha_C L_X + \beta_C L_C)\operatorname{Var}(\bar C_t)\\ \leq& B - \lambda\mathbb{E}H\left(\bar Z_t^i\right).\end{aligned}$$

The second inequality (4.2.7) is obtained exactly the same way.

The last inequality (4.2.8) simply relies on the sum of (4.2.5) for each i and the fact that $\mathcal{L}^{j,N}\left(H\left(Z_t^{i,N}\right)\right) = 0$ for $i \neq j$:

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\mathcal{L}^{N}\left(H\left(Z_t^{i,N}\right)\right)\\
&\quad=\frac{1}{N}\sum_{i=1}^{N}\mathcal{L}^{i,N}\left(H\left(Z_t^{i,N}\right)\right)\\
&\quad\leq\frac{1}{N}\sum_{i=1}^{N}\Bigg[B+(\alpha_X L_X+\beta_X L_C)\left(\left(\frac{1}{N}\sum_{k=1}^{N}|X_t^{k,N}|\right)^2-(X_t^{i,N})^2\right)\\
&\qquad+(\alpha_C L_X+\beta_C L_C)\left(\left(\frac{1}{N}\sum_{k=1}^{N}|C_t^{k,N}|\right)^2-(C_t^{i,N})^2\right)-\lambda H\left(Z_t^{i,N}\right)\Bigg]\\
&\quad\leq B-\lambda\frac{1}{N}\sum_{i=1}^{N}H\left(Z_t^{i,N}\right).
\end{aligned}
$$

The last inequality uses the fact that $\left(\frac{1}{N}\sum_{i=1}^{N}|y_i|\right)^2 - \frac{1}{N}\sum_{i=1}^{N}(y_i)^2 \leq 0$ for all $(y_i)_{1\leq i\leq N} \in \mathbb{R}^N$. $\square$

Bounds on the second moments of processes We can now prove the uniform in bounds on the second moments of $X_t^{i,N}$, $C_t^{i,N}$, $\bar{X}_t^i$ and $\bar{C}_t^i$.

Proof of Proposition 4.1.5. K_X and K_C are Lipschitz with constants L_X and L_C respectively. We do not assume any bound on these constants.
We assume for each $i \leq N$, $\mathbb{E}(|X_0^{i,N}|^2) < +\infty$ and $\mathbb{E}(|C_0^{i,N}|^2) < +\infty$.

The Inequation (4.2.7) gives us

$$\frac{d}{dt}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}H\left(Z_t^{i,N}\right)\right) \leq B - \lambda\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}H\left(Z_t^{i,N}\right)$$

for $\lambda \in \mathbb{R}$ which satisfies

$$\frac{L_X}{8} + L_C\left(2 + \frac{3}{32}\right) < 1 - \frac{\lambda}{2}.$$

For instance, let $\lambda = -2\left(\frac{L_X}{8} + L_C\left(2 + \frac{3}{32}\right)\right) < 0$. Then, using the Gronwall's lemma, we obtain

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}H\left(Z_t^{i,N}\right) - \frac{B}{\lambda} \leq e^{-\lambda t}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}H\left(Z_0^{i,N}\right) - \frac{B}{\lambda}\right).$$

Since Lemma 4.2.2 (i), i.e. $\mathbb{E}H\left(Z_t^{i,N}\right) \geq \frac{\gamma}{4}\mathbb{E}\left(|X_t^{i,N}|^2\right) + \frac{1}{4}\mathbb{E}\left(|C_t^{i,N}|^2\right)$ and $\mathbb{E}H\left(Z_0^{i,N}\right) \leq \gamma\mathbb{E}\left(|X_0^{i,N}|^2\right) + \mathbb{E}\left(|C_0^{i,N}|^2\right) + \frac{3}{2}H_0$, we obtain

$$\begin{aligned}
\frac{\gamma}{4N}\sum_{i=1}^{N}\mathbb{E}\left(|X_t^{i,N}|^2\right) + &\frac{1}{4N}\sum_{i=1}^{N}\mathbb{E}\left(|C_t^{i,N}|^2\right) \\
&\leq e^{-\lambda t}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}H\left(Z_0^{i,N}\right) - \frac{B}{\lambda}\right) + \frac{B}{\lambda} \\
&\leq \frac{e^{-\lambda t}}{N}\sum_{i=1}^{N}\left[\gamma\mathbb{E}\left(|X_0^{i,N}|^2\right) + \mathbb{E}\left(|C_0^{i,N}|^2\right) + \frac{3}{2}H_0\right] - e^{-\lambda t}\frac{\beta}{\lambda},
\end{aligned}$$

since $\beta/\lambda < 0$ with this choice of λ.

Then, there exists $\mathcal{C}_t < \infty$, such that for each $i \leq N$

$$\mathbb{E}\left(|X_t^{i,N}|^2 + |C_t^{i,N}|^2\right) \leq \mathcal{C}_t.$$

□

Proof of Proposition 4.1.6. K_X and K_C are Lipschitz with constants L_X and L_C respectively. We do not assume any bound on these constants.
We assume $\mathbb{E}(|\bar{X}_0^i|^2) < +\infty$ and $\mathbb{E}(|\bar{C}_0^i|^2) < +\infty$.

The proof is really similar to the last proof. The Inequation (4.2.6) gives us

$$\frac{d}{dt}\mathbb{E}H\left(\bar{Z}_t^i\right) \leq B - \lambda\mathbb{E}H\left(\bar{Z}_t^i\right).$$

for $\lambda \in \mathbb{R}$ which satisfies

$$\frac{L_X}{8} + L_C\left(2 + \frac{3}{32}\right) < 1 - \frac{\lambda}{2}.$$

For instance, let $\lambda = -2\left(\frac{L_X}{8} + L_C\left(2 + \frac{3}{32}\right)\right) < 0$. Then, using the Gronwall's lemma, we obtain

$$\mathbb{E}H\left(\bar{Z}_t^i\right) - \frac{B}{\lambda} \leq e^{-\lambda t}\left(\mathbb{E}H\left(\bar{Z}_0^i\right) - \frac{B}{\lambda}\right).$$

Thus, since $\beta/\lambda < 0$ with this choice of λ, we get

$$\frac{\gamma}{4}\mathbb{E}\left(|\bar{X}_t^i|^2\right) + \frac{1}{4}\mathbb{E}\left(|\bar{C}_t^i|^2\right) \leq e^{-\lambda t}\left[\gamma\mathbb{E}\left(|\bar{X}_0^i|^2\right) + \mathbb{E}\left(|\bar{C}_0^i|^2\right) + \frac{3}{2}H_0 - \frac{B}{\lambda}\right].$$

Finally

$$\mathbb{E}\left(|\bar{X}_t^i|^2 + |\bar{C}_t^i|^2\right) \leq C_{0,1}e^{C_{0,2}t}$$

with

$$C_{0,1} = \frac{4}{\max(\gamma, 1)} \left[\gamma \mathbb{E}\left(|\bar{X}_0^i|^2\right) + \mathbb{E}\left(|\bar{C}_0^i|^2\right) + \frac{3}{2} H_0 - \frac{B}{\lambda} \right],$$
$$C_{0,2} = -\lambda = 2\left(\frac{L_X}{8} + L_C\left(2 + \frac{3}{32}\right)\right).$$

□

In fact, when L_X and L_C are wisely bounded, the bound of the second moment can be uniform in time. It's the interest of the Lemma 4.2.5, which is proved below.

Proof of Lemma 4.2.5. We assume $\mathbb{E}(|\bar{X}_0^i|^2) < +\infty$ and $\mathbb{E}(|\bar{C}_0^i|^2) < +\infty$. Here, we also assume

$$\frac{L_X}{8} + L_C\left(2 + \frac{3}{32}\right) < 1.$$

Then, we can chose $\lambda > 0$ such that

$$\frac{L_X}{8} + L_C\left(2 + \frac{3}{32}\right) < 1 - \frac{\lambda}{2}.$$

The last proof gives us

$$\frac{\gamma}{4}\mathbb{E}\left(|\bar{X}_t^i|^2\right) + \frac{1}{4}\mathbb{E}\left(|\bar{C}_t^i|^2\right) \leq e^{-\lambda t}\left[\gamma \mathbb{E}\left(|\bar{X}_0^i|^2\right) + \mathbb{E}\left(|\bar{C}_0^i|^2\right) + \frac{3}{2}H_0 - \frac{B}{\lambda}\right] + \frac{B}{\lambda}.$$

If $\frac{B}{\lambda} < \gamma \mathbb{E}\left(|\bar{X}_0^i|^2\right) + \mathbb{E}\left(|\bar{C}_0^i|^2\right) + \frac{3}{2}H_0$, then we get

$$\begin{aligned}\frac{\gamma}{4}\mathbb{E}\left(|\bar{X}_t^i|^2\right) + \frac{1}{4}\mathbb{E}\left(|\bar{C}_t^i|^2\right) \leq& 1 \times \left[\gamma \mathbb{E}\left(|\bar{X}_0^i|^2\right) + \mathbb{E}\left(|\bar{C}_0^i|^2\right) + \frac{3}{2}H_0 - \frac{B}{\lambda}\right] + \frac{B}{\lambda} \\ \leq& \gamma \mathbb{E}\left(|\bar{X}_0^i|^2\right) + \mathbb{E}\left(|\bar{C}_0^i|^2\right) + \frac{3}{2}H_0,\end{aligned}$$

which is an uniform in time bound.

Else we get

$$\frac{\gamma}{4}\mathbb{E}\left(|\bar{X}_t^i|^2\right) + \frac{1}{4}\mathbb{E}\left(|\bar{C}_t^i|^2\right) \leq 0 + \frac{B}{\lambda},$$

which is also an uniform in time bound.

□

4.A.3 Proof of Lemma 4.2.9

We now prove that there are constants c, ϵ and δ such that

$$c + 2\epsilon\tilde{B} \leq \frac{\sigma_x^2}{2}\left(\int_0^R \Phi(s)\phi(s)^{-1}ds\right)^{-1} \tag{4.A.2}$$

$$2c + 4\epsilon\tilde{B} \leq \left(1 - L_C - \frac{1+L_X}{\delta}\right)\min_{r\in]0,R]}\frac{f'(r)r}{f(r)} \tag{4.A.3}$$

$$c \leq \frac{\lambda}{32}\frac{\frac{64\epsilon\tilde{B}}{15\lambda}}{1+\frac{64\epsilon\tilde{B}}{15\lambda}} \tag{4.A.4}$$

$$\delta > \frac{1+L_X}{1-L_C} \tag{4.A.5}$$

— Since for all $u \geq 0$, $0 < \phi(u) \leq 1$, we have $0 < \Phi(s) = \int_0^s \phi(u)\,du \leq s$, i.e $s/\Phi(s) \geq 1$. Therefore

$$\inf_{r\in]0,R]}\frac{r\phi(r)}{\Phi(r)} \geq \inf_{r\in]0,R]}\phi(r) = \phi(R).$$

It is thus sufficient for (4.A.3) that

$$2c + 4\epsilon\tilde{B} \leq \frac{1}{2}\left(1 - L_C - \frac{1+L_X}{\delta}\right)\phi(R).$$

— We have

$$\phi(r) \leq \exp\left(-\frac{1}{4\sigma_x^2}r^2\right).$$

So

$$\Phi(r) \leq \int_0^\infty \exp\left(-\frac{r^2}{4\sigma_x^2}\right)dr = \sigma_x\sqrt{\pi}.$$

Then

$$\int_0^R \frac{\Phi(r)}{\phi(r)}dr \leq \sigma_x\sqrt{\pi}R\frac{1}{\phi(R)}.$$

It is thus sufficient for (4.A.2) that

$$c + 2\epsilon\tilde{B} \leq \frac{\sigma_x}{2\sqrt{\pi}}\frac{\phi(R)}{R}.$$

— The various conditions involving c invite us to consider $2\epsilon\tilde{B} = \eta c$. Then

$$\begin{aligned} c \leq \frac{\lambda}{32}\frac{\frac{64\epsilon\tilde{B}}{15\lambda}}{1+\frac{64\epsilon\tilde{B}}{15\lambda}} &\iff c \leq \frac{\lambda}{32}\frac{32\eta c}{15\lambda + 32\eta c} \\ &\iff 1 \leq \lambda\frac{\eta}{15\lambda + 32\eta c} \text{ (since } c \geq 0) \\ &\iff c \leq \frac{\lambda}{32}\frac{\eta - 15}{\eta}. \end{aligned}$$

— We choose to write

$$\delta = (1+\tilde{\delta})\frac{1+L_{X,max}}{1-L_{C,max}} > \frac{1+L_X}{1-L_C}$$

— Let us assume, for simplicity, that $\epsilon \leq 1$. It is sufficient for this later condition to have

$$c \leq \frac{2\tilde{B}}{\eta}.$$

— The appearance of $\phi(R)$ suggests we should try to minimize it. We search for δ such that

$$\phi(r) \geq e^{-2} \exp\left(-(1+\gamma)\frac{L_X+\delta L_C+1}{4\sigma_x^2} r^2\right) \text{ on } [0,R].$$

We recall

$$\begin{aligned}\phi(r) &= \exp\left(-\frac{1}{4\sigma_x^2}\left(1+\delta\gamma+L_X+\delta L_C+\left(\epsilon\mathcal{C}_1^{(f)}+\mathcal{C}_2^{(f)}\right)\sigma_x^2\right)r^2\right)\\ &\geq \exp\left(-\frac{1}{4\sigma_x^2}\left(1+\delta\gamma+L_X+\delta L_C+\left(\mathcal{C}_1^{(f)}+\mathcal{C}_2^{(f)}\right)\sigma_x^2\right)r^2\right).\end{aligned}$$

It is therefore sufficient for (4.A.2) to have

$$c \leq \frac{1}{1+\eta}\frac{\sigma_x}{2\sqrt{\pi}}\frac{1}{R}\exp\left(-\frac{1}{4\sigma_x^2}\left(1+\delta\gamma+L_X+\delta L_C+\left(\mathcal{C}_1^{(f)}+\mathcal{C}_2^{(f)}\right)\sigma_x^2\right)R^2\right),$$

and for (4.A.3) to have

$$c \leq \frac{1}{2(1+\eta)}\frac{\tilde{\delta}(1-L_C)}{1+\tilde{\delta}}\exp\left(-\frac{1}{4\sigma_x^2}\left(1+\delta\gamma+L_X+\delta L_C+\left(\mathcal{C}_1^{(f)}+\mathcal{C}_2^{(f)}\right)\sigma_x^2\right)R^2\right).$$

— Finally, we bound L_X and L_C by either 0 or $L_{X,max}$ and $L_{C,max}$, to obtain bounds on c independent of L_X and L_C.

4.A.4 Proof of Lemma 4.2.10

Let $z, z' \in \mathbb{R}^2$.

Proof of control of the L^1 distance : We have

$$\|z-z'\|_1 = |x-x'|+|c-c'| \leq \frac{1}{\min(\delta,1)}\left(|x-x'|+\delta|c-c'|\right) = \frac{1}{\min(\delta,1)} r(z,z').$$

If $r(z,z') \leq 1 \leq R$, we have, using Lemma 4.2.9

$$r(z,z') \leq \frac{f(r)}{f'_-(R)} \leq \frac{f(r)}{\phi(R)g(R)}\left(1+\epsilon\tilde{H}(z)+\epsilon\tilde{H}(z')\right).$$

If $r(z,z') \geq 1$, we have, using (4.A.1)

$$\begin{aligned} r(z,z') \leq & r(z,z')^2 \\ \leq & \frac{16\epsilon(1+\delta^2)}{\min(\gamma,1)}\left(\epsilon H(z)+\epsilon H(z')\right) \\ \leq & \frac{16\epsilon(1+\delta^2)}{\min(\gamma,1)}\frac{f(r)}{f(1)}\left(1+\epsilon H(z)+\epsilon H(z')\right) \\ \leq & \frac{16\epsilon(1+\delta^2)}{\min(\gamma,1)}\frac{f(r)}{\phi(R)g(R)}\left(1+\epsilon \tilde{H}(z)+\epsilon \tilde{H}(z')\right). \end{aligned}$$

Thus

$$\|z-z'\|_1 \leq \frac{1}{\min(\delta,1)}\frac{1}{\phi(R)g(R)}\min\left(\frac{16\epsilon(1+\delta^2)}{\min(\gamma,1)},1\right) f(r(z,z'))\left(1+\epsilon \tilde{H}(z)+\epsilon \tilde{H}(z')\right).$$

Proof of control of the L2 distance : We have

$$\begin{aligned} r(z,z')^2 = & \left(|x-x'|+\delta|c-c'|\right)^2 \geq |x-x'|^2+\delta^2|c-c'|^2 \\ \geq & \min\left(1,\delta^2\right)\left(|x-x'|^2+|c-c'|^2\right). \end{aligned}$$

If $r(z,z') \geq 1$, we have, using (4.A.1)

$$\begin{aligned} r(z,z')^2 \leq & \frac{16\epsilon(1+\delta^2)}{\min(\gamma,1)}\left(\epsilon H(z)+\epsilon H(z')\right) \\ \leq & \frac{16\epsilon(1+\delta^2)}{\min(\gamma,1)}\frac{f(r)}{f(1)}\left(1+\epsilon H(z)+\epsilon H(z')\right) \\ \leq & \frac{16\epsilon(1+\delta^2)}{\min(\gamma,1)}\frac{f(r)}{\phi(R)g(R)}\left(1+\epsilon \tilde{H}(z)+\epsilon \tilde{H}(z')\right). \end{aligned}$$

If $r(z,z') \leq 1 \leq R$, we have, using Lemma 4.2.9

$$r(z,z')^2 \leq r(z,z') \leq \frac{f(r)}{f'_-(R)} \leq \frac{f(r)}{\phi(R)g(R)}\left(1+\epsilon \tilde{H}(z)+\epsilon \tilde{H}(z')\right).$$

Thus

$$\|z-z'\|_2^2 \leq \frac{1}{\min(\delta^2,1)}\frac{1}{\phi(R)g(R)}\min\left(\frac{16\epsilon(1+\delta^2)}{\min(\gamma,1)},1\right) f(r(z,z'))\left(1+\epsilon \tilde{H}(z)+\epsilon \tilde{H}(z')\right).$$

Proof of the second control of the L^1 distance : We have, if $r(z,z') \leq 1 \leq R$

$$r(z,z') \leq \frac{f(r)}{f'_-(R)} \leq \frac{f(r)}{\phi(R)g(R)}\left(1+\epsilon\sqrt{H(z)}+\epsilon\sqrt{H(z')}\right).$$

and, if $r(z,z') \geq 1$, recall Lemma 4.2.2.

$$\begin{aligned} \|z-z'\|_1 \leq & \sqrt{\frac{4}{\gamma}H(z)}+\sqrt{\frac{4}{\gamma}H(z')}+\sqrt{4H(z)}+\sqrt{4H(z')} \\ \leq & 4\max\left(\sqrt{\frac{1}{\gamma}},1\right)\left(\sqrt{H(z)}+\sqrt{H(z')}\right) \\ \leq & \frac{4}{\epsilon}\max\left(\sqrt{\frac{1}{\gamma}},1\right)\frac{f(r)}{\phi(R)g(R)}\left(1+\epsilon\sqrt{H(z)}+\epsilon\sqrt{H(z')}\right), \end{aligned}$$

and thus

$$\|z - z'\|_1 \leq \frac{1}{\phi(R)g(R)} \min\left(1, \frac{4}{\epsilon}\max\left(\sqrt{\frac{1}{\gamma}}, 1\right)\right) f(r(z,z'))\left(1 + \epsilon\sqrt{H(z)} + \epsilon\sqrt{H(z')}\right).$$

Independence with respect to L_X and L_C The *a priori* bounds $L_X \in [0, L_{X,max}]$ and $L_C \in [0, L_{C,max}]$ allow us to bound $\phi(R)$ independently of L_C and L_X by $\phi_{\min}$ (and we also use $g(R) \geq \frac{1}{2}$), thus giving us constant $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_z$ independent of L_C and L_X.

4.A.5 Proof of Lemmas 4.3.2 and 4.3.7

Proof of Lemma 4.3.2. Let's prove there exists an uniform in time bound on $\mathbb{E}(G^i_t)$ and $\mathbb{E}[(G^i_t)^2]$. First, let's recall the definition of G with Equation (4.2.27):

$$G^i_t = 1 + \epsilon\tilde{H}(\bar{Z}^i_t) + \epsilon\tilde{H}(Z^{i,N}_t) + \frac{\epsilon}{N}\sum_{j=1}^{N}\tilde{H}(Z^{j,N}_t) + \frac{\epsilon}{N}\sum_{j=1}^{N}\tilde{H}(\bar{Z}^j_t).$$

The idea is to bound the different expectation in terms of the expectation at time $t = 0$. Since $\mathbb{E}(e^{\tilde{a}(|X_0|+|C_0|)})$ is finite, we know that for each $k \in \mathbb{N}$, $\mathbb{E}(|X_0|^k)$ and $\mathbb{E}(|C_0|^k)$ are also finite. We deduce that for each $k \in \mathbb{N}$, for each $j \leq N$, $\mathbb{E}[H(\bar{Z}^j_0)^k]$ and $\mathbb{E}[H(Z^{j,N}_0)^k]$ are finite.

In fact, to bound uniformly in time the first moment, we only have to bound $\mathbb{E}(\tilde{H}(Z^{j,N}_t))$ and $\mathbb{E}(\tilde{H}(\bar{Z}^j_t))$ for each $j \leq N$. Let's begin with $\bar{Z}^j$. By (4.2.16), we have

$$\frac{d}{dt}\mathbb{E}\left[\tilde{H}\left(\bar{Z}^j_t\right)\right] \leq \tilde{B} - \frac{\lambda}{4}\mathbb{E}\left[\tilde{H}\left(\bar{Z}^j_t\right)\right],$$

then by Gronwall's lemma,

$$\begin{aligned}\mathbb{E}\left[\tilde{H}\left(\bar{Z}^j_t\right)\right] &\leq \frac{4\tilde{B}}{\lambda} + e^{-\frac{\lambda}{4}t}\left(\mathbb{E}\left[\tilde{H}\left(\bar{Z}^j_0\right)\right] - \frac{4\tilde{B}}{\lambda}\right)\\ &\leq \max\left(\mathbb{E}\left[\tilde{H}\left(\bar{Z}^j_0\right)\right], \frac{4\tilde{B}}{\lambda}\right).\end{aligned}$$

By Inequation (4.2.10) in the Lemma 4.2.6, we deduce the following inequation and we apply Cauchy-Schwarz

$$\begin{aligned}\mathbb{E}\left[\tilde{H}\left(\bar{Z}^j_0\right)\right] &\leq \mathbb{E}\left[H\left(\bar{Z}^j_0\right)\exp\left(a\sqrt{H\left(\bar{Z}^j_0\right)}\right)\right]\\ &\leq \mathbb{E}\left[H\left(\bar{Z}^j_0\right)^2\right]^{1/2}\mathbb{E}\left[\exp\left(2a\sqrt{H\left(\bar{Z}^j_0\right)}\right)\right]^{1/2} \qquad (4.A.6)\end{aligned}$$

We already know $\mathbb{E}\left[H\left(\bar{Z}_0^j\right)^2\right]$ is bounded. Now, it is enough to prove that there exist C such that for all $z \in \mathbb{R}^2$

$$\exp\left(2a\sqrt{H\left(z\right)}\right) \leq C \times e^{\tilde{a}(|x|+|c|)}.$$

In fact, from the definition of H in (4.2.2), we have

$$\begin{aligned}2\sqrt{H(z)} =& \sqrt{2}\sqrt{\gamma\left(x+\frac{\beta}{\gamma}\right)^2+(c+\alpha)^2+H_0}\\ \leq& \sqrt{2\gamma}\left|x+\frac{\beta}{\gamma}\right|+\sqrt{2}\left|c+\alpha\right|+\sqrt{H_0}\\ \leq& \sqrt{2\gamma}|x|+\sqrt{2}|c|+\frac{1}{a}\ln C,\end{aligned}$$

where C is a constant independent of z. Finally, since $\max\left(a\sqrt{2\gamma}, a\sqrt{2}\right) \leq \tilde{a}$, we have

$$\exp\left(2a\sqrt{H\left(z\right)}\right) \leq C \times e^{\tilde{a}(|x|+|c|)}.$$

Then, $\mathbb{E}\left[\exp\left(2a\sqrt{H\left(\bar{Z}_0^j\right)}\right)\right]$ is bounded and we deduce $\mathbb{E}(\tilde{H}(\bar{Z}_t^j))$ is bounded for each $j \leq N$ and all $t \geq 0$.

The same calculations can be done for $Z_t^{j,N}$. By (4.2.21), we have

$$\mathcal{L}^N\left(\frac{1}{N}\sum_{i=1}^N \tilde{H}(Z_t^{i,N})\right) \leq \tilde{B} - \frac{\lambda}{4}\left(\frac{1}{N}\sum_{i=1}^N \tilde{H}(Z_t^{i,N})\right).$$

In particular,

$$\frac{d}{dt}\left[\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^N \tilde{H}(Z_t^{i,N})\right)\right] \leq \mathbb{E}\left[\mathcal{L}^N\left(\frac{1}{N}\sum_{i=1}^N \tilde{H}(Z_t^{i,N})\right)\right] \leq \tilde{B} - \frac{\lambda}{4}\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^N \tilde{H}(Z_t^{i,N})\right),$$

and we can use Gronwall's lemma. The following is exactly as above.

Finally, we have proved that for each $j \leq N$, $\mathbb{E}(\tilde{H}(Z_t^{j,N}))$ and $\mathbb{E}(\tilde{H}(\bar{Z}_t^j))$ are bounded uniformly in time . Thus, $\mathbb{E}(G_t^i)$ is bounded uniformly in time.

To bound the second moment of G_t^i, we have to bound each type of the following expectations: $\mathbb{E}[\tilde{H}(Z_t^{j_1,N})\tilde{H}(Z_t^{j_2,N})]$, $\mathbb{E}[\tilde{H}(Z_t^{j_1,N})\tilde{H}(\bar{Z}_t^{j_2})]$, $\mathbb{E}[\tilde{H}(\bar{Z}_t^{j_1})\tilde{H}(\bar{Z}_t^{j_2})]$, $\mathbb{E}[\tilde{H}(Z_t^{j,N})^2]$ and $\mathbb{E}[\tilde{H}(\bar{Z}_t^j)^2]$. By Cauchy-Schwarz, it is in fact enough to bound $\mathbb{E}[\tilde{H}(Z_t^{j,N})^2]$ and $\mathbb{E}[\tilde{H}(\bar{Z}_t^j)^2]$.

First, by the definition of $\tilde{H}$ in (4.2.9) in Subsection 4.2.3,

$$\begin{aligned}\tilde{H}(z)^2 =& \left(\frac{2}{a^2}\exp\left(a\sqrt{H(z)}\right)\left(a\sqrt{H(z)}-1\right)+\frac{2}{a^2}\right)^2\\ \leq& 2\frac{2^2}{a^4}\exp\left(2a\sqrt{H(z)}\right)\left(a\sqrt{H(z)}-1\right)^2+2\frac{2^2}{a^4}\\ \leq& \frac{8}{a^2}\exp\left(2a\sqrt{H(z)}\right)\left(2a^2H(z)+2\right)+\frac{8}{a^2}.\end{aligned}$$

As for the first moment, the study of $Z_t^{j,N}$ is very similar to the one of $\bar{Z}_t^j$. Here, we only focus on the second one.

To bound $\mathbb{E}[\tilde{H}(\bar{Z}_t^j)^2]$, we only have to bound $\mathbb{E}\left[\exp\left(2a\sqrt{H(\bar{Z}_t^j)}\right)H(\bar{Z}_t^j)\right]$ and $\mathbb{E}\left[\exp\left(2a\sqrt{H(\bar{Z}_t^j)}\right)\right]$. By Cauchy-Schwarz,

$$\mathbb{E}\left[\exp\left(2a\sqrt{H(\bar{Z}_t^j)}\right)H(\bar{Z}_t^j)\right]\leq\mathbb{E}\left[\exp\left(4a\sqrt{H(\bar{Z}_t^j)}\right)\right]^{1/2}\mathbb{E}\left[H(\bar{Z}_t^j)^2\right]^{1/2}.$$

First, we know $\mathbb{E}(\tilde{H}(\bar{Z}_t^j))$ is bounded uniformly in time. By Inequation (4.2.11) from Lemma 4.2.6, we have

$$\sqrt{H(z)}\exp\left(a\sqrt{H(z)}\right)\leq a\tilde{H}(z)+\frac{1}{a}(e-2).$$

Then $\mathbb{E}\left(\sqrt{H(\bar{Z}_t^j)}\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)\right)$ is bounded uniformly in time. In particular, each moment of $\sqrt{H(\bar{Z}_t^j)}$ is uniformly bounded in time and it is the case for $H(\bar{Z}_t^j)^2$. Now, it remains to bound $\mathbb{E}\left[\exp\left(4a\sqrt{H(\bar{Z}_t^j)}\right)\right]$. First, let's notice for all $z\in\mathbb{R}^2$, $2H(z)\leq H(2z)+H_0$. Then, by (4.2.10) from Lemma 4.2.6, we know

$$\forall z\in\mathbb{R}^2,\exp\left(4a\sqrt{H(z)}\right)\leq\exp\left(a\sqrt{H(2z)}\right)\leq\tilde{H}(2z)+\mathcal{C},$$

where $\mathcal{C}$ is a constant. Let's denote $\tilde{G}(z)=\tilde{H}(2z)$. A study on the generator shows that

$$\forall z\in\mathbb{R}^2,\mathcal{L}_\mu\tilde{G}(z)\leq 4\mathcal{L}_\mu\tilde{H}(z).$$

Thus, by Dynkin's formula,

$$\begin{aligned}\mathbb{E}(\tilde{G}(\bar{Z}_t^j))\leq&\mathbb{E}(\tilde{G}(\bar{Z}_0^j))+\mathbb{E}\left[\int_0^t\mathcal{L}_\mu\tilde{G}(\bar{Z}_s^j)ds\right]\\ \leq&\mathbb{E}(\tilde{G}(\bar{Z}_0^j))+4\mathbb{E}\left[\int_0^t\mathcal{L}_\mu\tilde{H}(\bar{Z}_s^j)ds\right]\\ \leq&\mathbb{E}(\tilde{G}(\bar{Z}_0^j))+4\tilde{B}-\lambda\int_0^t\mathbb{E}\left[\mathcal{L}_\mu\tilde{H}(\bar{Z}_s^j)\right]ds,\end{aligned}$$

where we have used (4.2.10) in (4.2.14) for the last inequality. Since $\mathcal{L}_\mu \tilde{H}$ is a positive function, we have:

$$\mathbb{E}(\tilde{G}(\bar{Z}_t^j)) \leq \mathbb{E}(\tilde{G}(\bar{Z}_0^j)) + 4\tilde{B} \leq \mathbb{E}(\tilde{H}(2\bar{Z}_0^j)) + 4\tilde{B}.$$

We can make the same decomposition by Cauchy-Schwarz than above in (4.A.6). Then, since H is a polynome, $\mathbb{E}\left[H(2\bar{Z}_0^j)^2\right]$ is also bounded uniformly in time. Since $2\max(a\sqrt{2\gamma}, a\sqrt{2}) \leq \tilde{a}$, we have $\mathbb{E}\left[\exp\left(2a\sqrt{H\left(2\bar{Z}_0^j\right)}\right)\right]$ is bounded. Thus we deduce $\mathbb{E}(\tilde{H}(\bar{Z}_t^j)^2)$ is bounded uniformly in time.

Finally, we deduce $\mathbb{E}\left((G_t^i)^2\right)$ is bounded uniformly in time. □

Proof of Lemma 4.3.7. Using $\partial_x H(z) = \gamma x + \beta$, we have

$$\begin{aligned}
&\left|\partial_x \tilde{H}(Z_t^{i,N}) - \partial_x \tilde{H}(\bar{Z}_t^i)\right| \\
&= \left|\left(\gamma X_t^{i,N} + \beta\right)\exp\left(a\sqrt{H(Z_t^{i,N})}\right) - (\gamma \bar{X}_t^i + \beta)\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right| \\
&\leq \left|\gamma X_t^{i,N} - \gamma \bar{X}_t^i\right|\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right) + \exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right) \\
&\quad + \left|\gamma \bar{X}_t^i + \beta\right|\left|\exp\left(a\sqrt{H(Z_t^{i,N})}\right) - \exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right|.
\end{aligned}$$

Since $\left|X_t^{i,N} - \bar{X}_t^i\right| \leq r_t^i$,

$$\begin{aligned}
&\left|\gamma X_t^{i,N} - \gamma \bar{X}_t^i\right|\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right) + \exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right) \\
&\qquad\qquad \leq \gamma r_t^i\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right) + \exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right).
\end{aligned}$$

By Lemma 4.2.2 (ii), we have $H(z) \geq \frac{1}{2}\min\left(\frac{1}{\gamma}, 1\right)(\gamma x + \beta)^2$. By the mean value theorem, for all $y_1 \leq y_2$ in $\mathbb{R}$, there exists $y_3 \in [y_1, y_2]$ such that: $e^{ay_1} - e^{ay_2} = a(y_1 - y_2)e^{ay_3}$. In particular, $|e^{ay_1} - e^{ay_2}| \leq a|y_1 - y_2|(e^{ay_1} + e^{ay_2})$. Thus

$$\begin{aligned}
&\left|\gamma \bar{X}_t^i + \beta\right|\left|\exp\left(a\sqrt{H(Z_t^{i,N})}\right) - \exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right| \\
&\leq a\sqrt{\frac{2H(\bar{Z}_t^i)}{\min\left(\frac{1}{\gamma}, 1\right)}}\left|\sqrt{H(Z_t^{i,N})} - \sqrt{H(\bar{Z}_t^i)}\right|\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right. \\
&\qquad\qquad \left. + \exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right) \\
&\leq a\sqrt{2\max(\gamma, 1)}\left|H(Z_t^{i,N}) - H(\bar{Z}_t^i)\right|\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right) + \exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right).
\end{aligned}$$

Then by the definition of H we get

$$\begin{aligned}\left|H(Z_t^{i,N})-H(\bar{Z}_t^i)\right| =&\left|\frac{1}{2}\gamma\left((X_t^{i,N})^2-(\bar{X}_t^i)^2\right)+\beta(X_t^{i,N}-\bar{X}_t^i)\right.\\ &\left.+\frac{1}{2}\left((C_t^{i,N})^2-(\bar{C}_t^i)^2\right)+\alpha(C_t^{i,N}-\bar{C}_t^i)\right|\\ \leq&\frac{1}{2}\gamma\left|X_t^{i,N}-\bar{X}_t^i\right|\left|X_t^{i,N}+\bar{X}_t^i\right|+\beta\left|X_t^{i,N}-\bar{X}_t^i\right|\\ &+\frac{1}{2}\left|C_t^{i,N}-\bar{C}_t^i\right|\left|C_t^{i,N}+\bar{C}_t^i\right|+\alpha\left|C_t^{i,N}-\bar{C}_t^i\right|.\end{aligned}$$

Now, by Lemma 4.2.2 (i), we have $H(z)\geq\frac{\gamma}{4}x^2+\frac{1}{4}c^2$ and since $\left|X_t^{i,N}-\bar{X}_t^i\right|\leq r_t^i$ and $\left|C_t^{i,N}-\bar{C}_t^i\right|\leq r_t^i/\delta$, we get

$$\left|X_t^{i,N}-\bar{X}_t^i\right|\left(\frac{1}{2}\gamma\left|X_t^{i,N}+\bar{X}_t^i\right|+\beta\right)\leq r_t^i\left(\sqrt{\gamma}\left(\sqrt{H(Z_t^{i,N})}+\sqrt{H(\bar{Z}_t^i)}\right)+\beta\right)$$

and

$$\left|C_t^{i,N}-\bar{C}_t^i\right|\left(\frac{1}{2}\left|C_t^{i,N}+\bar{C}_t^i\right|+\alpha\right)\leq\frac{r_t^i}{\delta}\left(\sqrt{H(Z_t^{i,N})}+\sqrt{H(\bar{Z}_t^i)}+\alpha\right).$$

Thus

$$\left|H(Z_t^{i,N})-H(\bar{Z}_t^i)\right|\leq\left(\beta+\frac{\alpha}{\delta}\right)r_t^i+\left(\sqrt{\gamma}+\frac{1}{\delta}\right)r_t^i\left(\sqrt{H(Z_t^{i,N})}+\sqrt{H(\bar{Z}_t^i)}\right).$$

Finally,

$$\begin{aligned}&|\partial_x\tilde{H}(Z_t^{i,N})-\partial_x\tilde{H}(\bar{Z}_t^i)|\\ &\leq\gamma r_t^i\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right)+\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right)\\ &\quad+a\sqrt{2\max(\gamma,1)}\left(\beta+\frac{\alpha}{\delta}\right)r_t^i\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right)+\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right)\\ &\quad+a\sqrt{2\max(\gamma,1)}\left(\sqrt{\gamma}+\frac{1}{\delta}\right)r_t^i\left(\sqrt{H(Z_t^{i,N})}+\sqrt{H(\bar{Z}_t^i)}\right)\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right.\\ &\quad\left.+\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right)\\ &\leq r_t^i\left(\gamma+a\sqrt{2\max(\gamma,1)}\left(\beta+\frac{\alpha}{\delta}\right)\right)\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right)+\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right)\\ &\quad+ar_t^i\sqrt{2\max(\gamma,1)}\left(\sqrt{\gamma}+\frac{1}{\delta}\right)\left(2\sqrt{H(Z_t^{i,N})}\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right.\\ &\quad\left.+2\sqrt{H(\bar{Z}_t^i)}\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right).\end{aligned}$$

Now, we can finally use Lemma 4.2.6, and more precisely (4.2.10) and (4.2.11), we obtain

$$
\begin{aligned}
&|\partial_x \tilde{H}(Z_t^{i,N}) - \partial_x \tilde{H}(\bar{Z}_t^i)| \\
&\le r_t^i \left(\gamma + a\sqrt{2\max(\gamma,1)}\left(\beta + \frac{\alpha}{\delta}\right)\right)\left(\tilde{H}(Z_t^{i,N}) + \tilde{H}(\bar{Z}_t^i) + \frac{4}{a^2}\left(\exp\left(\frac{a^2}{2}\right) - 1\right)\right) \\
&\quad + a r_t^i \sqrt{2\max(\gamma,1)}\left(\sqrt{\gamma} + \frac{1}{\delta}\right)\left(2a\tilde{H}(Z_t^{i,N}) + \frac{2}{a}(e-2) + 2a\tilde{H}(\bar{Z}_t^i) + \frac{2}{a}(e-2)\right) \\
&\le r_t^i \left(\tilde{H}(Z_t^{i,N}) + \tilde{H}(\bar{Z}_t^i)\right)\left[\gamma + a\sqrt{2\max(\gamma,1)}\left(\beta + \frac{\alpha}{\delta}\right) + 2a^2\sqrt{2\max(\gamma,1)}\left(\sqrt{\gamma} + \frac{1}{\delta}\right)\right] \\
&\quad + r_t^i \left[\left(\gamma + a\sqrt{2\max(\gamma,1)}\left(\beta + \frac{\alpha}{\delta}\right)\right)\frac{4}{a^2}\left(\exp\left(\frac{a^2}{2}\right) - 1\right)\right. \\
&\quad \left. + 4\sqrt{2\max(\gamma,1)}\left(\sqrt{\gamma} + \frac{1}{\delta}\right)(e-2)\right].
\end{aligned}
$$

We denote by $\mathcal{C}_1^{(f)}$ and $\mathcal{C}_2^{(f)}$ (given in Assumption 4.2.8) the following constants

$$
\begin{aligned}
\mathcal{C}_1^{(f)} &= 4\left[\left(\gamma + a\sqrt{2\max(\gamma,1)}\left(\beta + \frac{\alpha}{\delta}\right)\right)\frac{4}{a^2}\left(\exp\left(\frac{a^2}{2}\right) - 1\right)\right. \\
&\quad \left. + 4\sqrt{2\max(\gamma,1)}\left(\sqrt{\gamma} + \frac{1}{\delta}\right)(e-2)\right] \\
\mathcal{C}_2^{(f)} &= 4\left[\gamma + a\sqrt{2\max(\gamma,1)}\left(\beta + \frac{\alpha}{\delta}\right) + 2a^2\sqrt{2\max(\gamma,1)}\left(\sqrt{\gamma} + \frac{1}{\delta}\right)\right]
\end{aligned}
$$

By the definition of G_t^i and since $G_t^i \ge 1$, we obtain

$$
|\partial_x \tilde{H}(Z_t^{i,N}) - \partial_x \tilde{H}(\bar{Z}_t^i)| \le r_t^i \frac{G_t^i}{\epsilon}\frac{\mathcal{C}_2^{(f)}}{4} + r_t^i G_t^i \frac{\mathcal{C}_1^{(f)}}{4},
$$

and eventually

$$
\begin{aligned}
&2\epsilon\left(1 + \frac{1}{N}\right)\sigma_x^2 \varphi_{\mathrm{rc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right)^2 \left|\partial_x \tilde{H}(Z_t^{i,N}) - \partial_x \tilde{H}(\bar{Z}_t^i)\right| \\
&\qquad\qquad \le \left(\epsilon \mathcal{C}_1^{(f)} + \mathcal{C}_2^{(f)}\right)\sigma_x^2 \varphi_{\mathrm{rc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right)^2 r_t^i G_t^i.
\end{aligned}
$$

□

4.B Proof of Theorem 4.1.4 in the case $\sigma_X = 0$ and $\sigma_C > 0$

We quickly explain in this section how we may also deal with the case $\sigma_X = 0$ and $\sigma_C > 0$. Recall how the choice of the coupling method was motivated by the observation in (4.1.8) that the difference of potentials $\left|C_t^{i,N} - \bar{C}_t^i\right|$ was naturally contracting when $\left|X_t^{i,N} - \bar{X}_t^i\right|$ was close to 0. This lead us to using a reflection coupling on the Brownian motions acting on the potential X, to bring the difference close to 0, and it was thus necessary for σ_X to be positive (σ_C however did not hold any importance). In the case $\sigma_X = 0$, we then have to assume $\sigma_C > 0$, and we do a change of variable, motivated by the following observation. We have, when $\sigma_X = 0$

$$\begin{aligned} d(X_t^i - \bar{X}_t^i) &= \left((X_t^i - \bar{X}_t^i) - ((X_t^i)^3 - (\bar{X}_t^i)^3) - (C_t^i - \bar{C}_t^i)\right) dt \\ &\quad + \left(\frac{1}{N} \sum_{j=1}^{N} K_X(Z_t^i - Z_t^j) - K_X * \bar{\rho}(\bar{Z}_t^i) \right) dt, \\ &= \left(2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i) - (X_t^i - \bar{X}_t^i) - ((X_t^i)^3 - (\bar{X}_t^i)^3)\right) dt \\ &\quad + \left(\frac{1}{N} \sum_{j=1}^{N} K_X(Z_t^i - Z_t^j) - K_X * \bar{\rho}(\bar{Z}_t^i) \right) dt. \end{aligned}$$

Thus

$$\begin{aligned} d|X_t^i - \bar{X}_t^i| &= \left(\operatorname{sign}(X_t^i - \bar{X}_t^i) \left(2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)\right) - |(X_t^i)^3 - (\bar{X}_t^i)^3| \right. \\ &\quad \left. - |X_t^i - \bar{X}_t^i|\right) dt + \operatorname{sign}(X_t^i - \bar{X}_t^i) \left(\frac{1}{N} \sum_{j=1}^{N} K_X(Z_t^i - Z_t^j) - K_X * \bar{\rho}(\bar{Z}_t^i) \right) dt. \end{aligned}$$

The quantity $|X_t^i - \bar{X}_t^i|$ is therefore naturally contracting when $|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)|$ is close to 0. Thanks to the presence of a Brownian motion in the stochastic differential equations defining the potential C, we can now use a reflection coupling to have $|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)|$ go to 0. Consider the following coupling

$$\begin{cases} dX_t^i = & (X_t^i - (X_t^i)^3 - C_t^i - \alpha)dt + \frac{1}{N} \sum_{j=1}^N K_X(Z_t^i - Z_t^j)dt \\ dC_t^i = & (\gamma X_t^i - C_t^i + \beta)dt + \frac{1}{N} \sum_{j=1}^N K_C(Z_t^i - Z_t^j)dt \\ & +\sigma_c \varphi_{\mathrm{sc}} \left(|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)|\right) dB_t^{i,sc,C} \\ & +\sigma_c \varphi_{\mathrm{rc}} \left(|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)|\right) dB_t^{i,rc,C} \end{cases} \tag{4.B.1}$$

and

$$\begin{cases} d\bar{X}_t^i = & (\bar{X}_t^i - (\bar{X}_t^i)^3 - \bar{C}_t^i - \alpha)dt + K_X * \bar{\rho}(\bar{Z}_t^i)dt \\ d\bar{C}_t^i = & (\gamma \bar{X}_t^i - \bar{C}_t^i + \beta)dt + K_C * \bar{\rho}(\bar{Z}_t^i)dt \\ & +\sigma_c \varphi_{\mathrm{sc}} \left(|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)|\right) dB_t^{i,sc,C} \\ & -\sigma_c \varphi_{\mathrm{rc}} \left(|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)|\right) dB_t^{i,rc,C} \\ \bar{\rho} = & \mathcal{L}((\bar{X}_t^1, \bar{C}_t^1)) \end{cases}, \tag{4.B.2}$$

and for $\delta > 0$, the following modified distance $r_t^i = \delta|X_t^i - \bar{X}_t^i| + |2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)|$. Like previously, we consider a modified semimetric of the form $\frac{1}{N} \sum f(r_t^i) G_t^i$, and similar calculations yield

$$d(e^{ct} f(r_t^i) G_t^i) \leq e^{ct} K_t^i dt + dM_t^i,$$

where M_t^i is a continuous local martingale and

$$K_t^i = \tilde{K}_t^i + I_t^{1,i} + I_t^{2,i} + I_t^{3,i}.$$

We define $\tilde{K}_t^i$, $I_t^{1,i}$, $I_t^{2,i}$, $I_t^{3,i}$ and $I_t^{4,i}$ as follows:

$$\begin{aligned}
\tilde{K}_t^i =& G_t^i \Big[2cf(r_t^i) + 2f''(r_t^i)\sigma_c^2 \varphi_{\mathrm{rc}} \left(|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)| \right)^2 \\
&+ f'(r_t^i) \left(\left|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)\right| (\delta+1) - |(X_t^i)^3 - (\bar{X}_t^i)^3|(\delta - 2) \right. \\
&+ |X_t^i - \bar{X}_t^i| \left(-\delta + \gamma + L_X(\delta+2) + L_C\right) + |C_t^i - \bar{C}_t^i|(L_X(\delta+2) + L_C) \\
&+ \sigma_c^2 \varphi_{\mathrm{rc}} \left(|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)| \right)^2 \left(\epsilon \mathcal{C}_1^{(f)} + \mathcal{C}_2^{(f)} \right) r_t^i \Big) \Big] \\
&+ \epsilon f(r_t^i) \left(4\tilde{B} - \frac{\lambda}{8} \tilde{H}(\bar{Z}_t^i) - \frac{\lambda}{8} \tilde{H}(Z_t^{i,N}) - \frac{\lambda}{8N} \sum_{j=1}^N \tilde{H}(\bar{Z}_t^j) - \frac{\lambda}{32N} \sum_{j=1}^N \tilde{H}(Z_t^{j,N}) \right),
\end{aligned}$$

$$\begin{aligned}
I_t^{1,i} =& G_t^i f'(r_t^i) \left[(\delta+2) \left(\left| \frac{1}{N} \sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i) \right| \right) \right. \\
&+ \left. \left(\left| \frac{1}{N} \sum_{j=1}^N K_C(\bar{Z}_t^i - \bar{Z}_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i) \right| \right) \right],
\end{aligned}$$

$$\begin{aligned}
I_t^{2,i} =& G_t^i f'(r_t^i) \left[(\delta+2) \left(\frac{L_X}{N} \left(\sum_{j=1}^N |X_t^{j,N} - \bar{X}_t^j| + |C_t^{j,N} - \bar{C}_t^j| \right) \right) \right. \\
&+ \left. \left(\frac{L_C}{N} \left(\sum_{j=1}^N |X_t^{j,N} - \bar{X}_t^j| + |C_t^{j,N} - \bar{C}_t^j| \right) \right) \right] - cf(r_t^i) G_t^i \\
&- \epsilon f(r_t^i) \left[\frac{\lambda}{32} H(\bar{Z}_t^i) \exp\left(a\sqrt{H(\bar{Z}_t^i)} \right) + \frac{\lambda}{32} H(Z_t^{i,N}) \exp\left(a\sqrt{H(Z_t^{i,N})} \right) \right. \\
&+ \left. \frac{\lambda}{32N} \sum_{j=1}^N H(\bar{Z}_t^j) \exp\left(a\sqrt{H(\bar{Z}_t^j)} \right) + \frac{\lambda}{32N} \sum_{j=1}^N H(Z_t^{j,N}) \exp\left(a\sqrt{H(Z_t^{j,N})} \right) \right],
\end{aligned}$$

$$\begin{aligned}
I_t^{3,i} =& \epsilon f(r_t^i) \left((\alpha_X L_X + \beta_X L_C) \left(\frac{\sum_{j=1}^N |X_t^{j,N}|}{N} \right)^2 \exp\left(a\sqrt{H(Z_t^{i,N})} \right) \right. \\
&+ (\alpha_C L_X + \beta_C L_C) \left(\frac{\sum_{j=1}^N |C_t^{j,N}|}{N} \right)^2 \exp\left(a\sqrt{H(Z_t^{i,N})} \right) \\
&- \left. \frac{\lambda}{16} H(Z_t^{i,N}) \exp\left(a\sqrt{H(Z_t^{i,N})} \right) - \frac{\lambda}{16N} \sum_{j=1}^N H(Z_t^{j,N}) \exp\left(a\sqrt{H(Z_t^{j,N})} \right) \right).
\end{aligned}$$

We then have the additional constraint of $\delta > 2$ (so that the coefficient appearing in front of $|(X_t^i)^3 - (\bar{X}_t^i)^3|$ in the expression of $\tilde{K}_t^i$ is non positive). Otherwise, we deal with the various terms exactly as previously, through the choice of a sufficiently concave function f and a law of large numbers, and by considering the regions of space

$$\begin{aligned}
\mathrm{Reg}_1^i &= \left\{(\bar{Z}_t^i, Z_t^{i,N}) \text{ s.t. } |2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)| \geq \xi \text{ and } r_t^i \leq R\right\}, \\
\mathrm{Reg}_2^i &= \left\{(\bar{Z}_t^i, Z_t^{i,N}) \text{ s.t. } |2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)| < \xi \text{ and } r_t^i \leq R_1\right\}, \\
\mathrm{Reg}_3^i &= \left\{(\bar{Z}_t^i, Z_t^{i,N}) \text{ s.t. } r_t^i > R\right\}.
\end{aligned}$$

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Comportement en temps long de différents processus stochastiques en neuroscience

Résumé: Dans cette thèse, nous nous concentrons sur deux modèles stochastiques pouvant être appliqués aux neurosciences : le modèle de Hawkes et le modèle de FitzHugh-Nagumo. Nous étudions leur comportement en temps long.

Le premier chapitre porte sur les processus cumulatifs, qui sont une classe de processus plus généraux que les processus de renouvellement. Ces processus cumulent des variables aléatoires indépendantes au cours du temps. Ces variables aléatoires sont ajoutées sur des intervalles de temps donnés par un processus de renouvellement. En nous inspirant des travaux de Lefevere, Mariani et Zambotti (2011), nous démontrons un Principe de Grandes Déviations pour ces processus, ainsi que des inégalités de grandes déviations dans un cadre plus général.

Le second chapitre est dédié aux processus de Hawkes, dans un contexte non linéaire, avec une fonction de reproduction signée. Ils permettent ainsi de modéliser de l'auto-excitation et de l'auto-inhibition. Nous prouvons une loi des grandes nombres, un théorème central limite et des résultats de grandes déviations pour un processus de Hawkes (unique). Ces résultats reposent sur une structure de renouvellement pour ces processus, introduite par Costa, Graham, Marsalle et Tran (2020), qui permettent de définir les processus de Hawkes comme des processus cumulatifs. Nous utilisons alors des résultats déjà connus pour les processus cumulatifs et les résultats obtenus dans le chapitre 1. Nous exhibons également deux exemples dans lesquels des calculs explicites sont faits.

Le dernier chapitre est un travail effectué en collaboration avec Pierre Le Bris et est consacré à l'étude de plusieurs processus de FitzHugh-Nagumo stochastiques en interaction. La spécificité de ce modèle défini par des Equations Différentielles Stochastiques est son terme cubique dans la dérive, qui est donc non-Lipschitz. Nous nous intéressons au cadre d'interactions champ moyen, et nous montrons une propagation du chaos, d'abord non-uniforme en temps puis uniforme en temps. Pour ce faire, nous utilisons une méthode de couplage mixte, c'est-à-dire un couplage synchrone sur un certain sous-espace et un couplage symétrique sur l'espace complémentaire. Nous exhibons également des bornes explicites pour ces résultats.

On asymptotic behaviour of stochastic processes on neuroscience

Abstract: In this thesis, we focus on two stochastic models which can be applied to neuroscience : Hawkes model and FitzHugh-Nagumo model. We study their long-time behavior.

The first chapter deals with cumulative processes, which are a larger processes class than renewal processes. These processes accumulate independent random variables over time. These random variables are added on time intervals given by a renewal process. Inspired by the work of Lefevere, Mariani and Zambotti (2011), we prove a Large Deviations Principle for these processes, and large deviations inequalities in a more general framework.

The second chapter is dedicated to Hawkes processes, in a non-linear context, with a signed reproduction function. They model self-excitation and self-inhibition. We prove a law of large numbers, a central limit theorem and large deviations results for a unique Hawkes process. These results lie on a renewal structure for these processes introduced by Costa, Graham, Marsalle and Tran (2020), which leads to a comparison with cumulative processes. Thus, we use known results for cumulative processes and results obtained in Chapter 1. We also exhibit two examples with explicit computations.

The last chapter is a joint work with Pierre Le Bris and is devoted to the study of stochastic FitzHugh-Nagumo processes in interaction. The specificity of this model, described by Stochastic Differential Equations, is its cubic term in the drift which is non-Lipschitz. We focus on mean-field interactions and we prove a propagation of chaos, non-uniform in time first, and then a uniform in time one. To do so, we use a combined coupling method, i.e. a synchronous coupling on a specific subspace and an antithetic coupling on the complementary subspace. We also exhibit explicit bounds for these results.